SCUOLA NAZIONALE DOTTORANDI DI ELETTROTECNICA
“FERDINANDO GASPARINI”

XXVII Stage

Introduction to Circuit Quantum Electrodynamics

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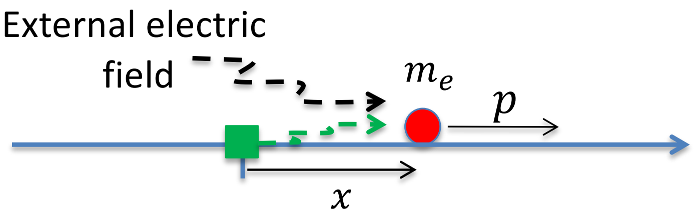
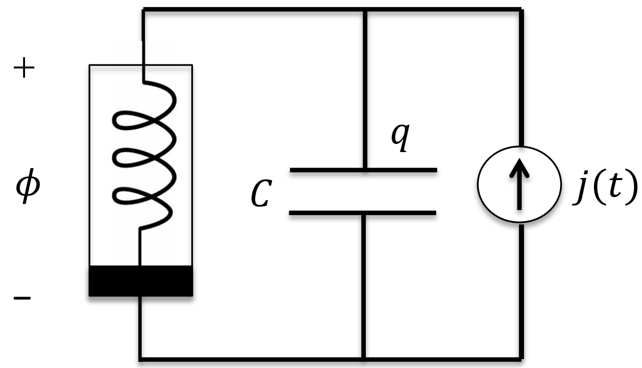
Lecture Outline

2.1.1 A Look to Quantum Mechanics: Some fundamental concepts

C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. I, 2nd ed., Wiley-VCH, 2019.

Electron in 1D Potential Well Versus Superconducting Circuit





Electron in a 1D potential well under the action of an external time-varying electric field

Superconducting nonlinear circuit driven by a current source

Canonically conjugate variables (x, p)

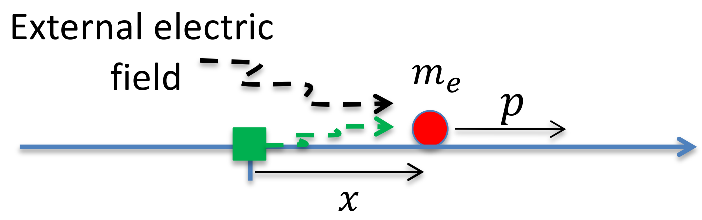
Hamiltonian $H(p, x; t) = \frac{1}{2m_e}p^2 + W_a(x) - F(t)x$

Canonically conjugate variables (ϕ, q)

Hamiltonian $H(q, \phi; t) = \frac{1}{2C}q^2 + W_J(\phi) - j(t)\phi$

We now introduce fundamental concepts of quantum mechanics by referring to the electron in 1D potential well, the extension to the superconducting circuit with one degree of freedom is immediate

Physical Quantities



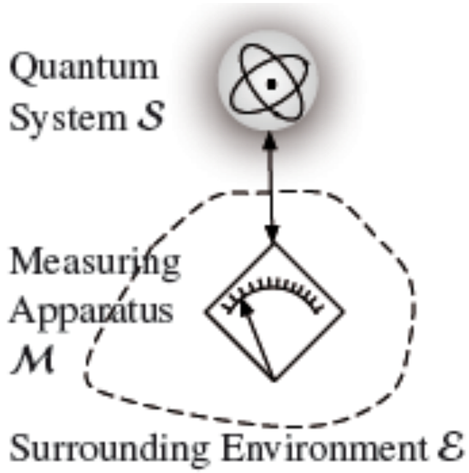
Physical Quantities are properties that can be measured, in principle, with infinite precision:

- particle position;
- particle linear momentum;
- particle energy;
- ….

Measurement in Quantum Mechanics

"By measurement, in quantum mechanics, we understand any process of interaction between classical objects (*apparatus*) and quantum objects, occurring apart from and independently of any observer."
(L. D. Landau, E. M. Lifshitz, **Quantum Mechanics**).

The *quantum system* and the *measuring apparatus* form an **isolated system**.



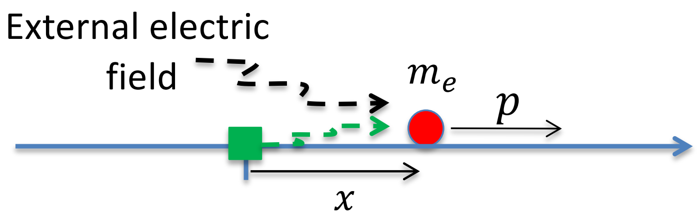
Conservation of *linear momentum*, *angular momentum* and *energy* also holds for such a system: it is only consequence of *time homogeneity*, *space homogeneity* and *space isotropy*, respectively.

The changes in linear momentum, angular momentum and energy of the measuring apparatus allows one to infer the change in linear momentum, angular momentum and energy of micro - particles.

Classical Mechanics

In **classical mechanics** the values that physical quantities assume belong to continuum sets.

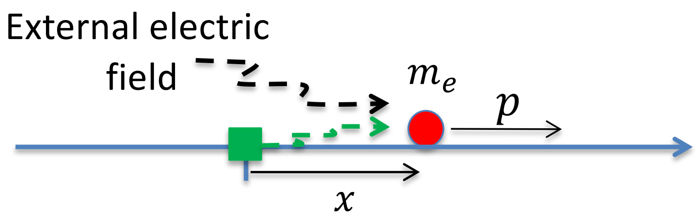
Quantum Mechanics



In **quantum mechanics** the values of physical quantities can belong to discrete sets.

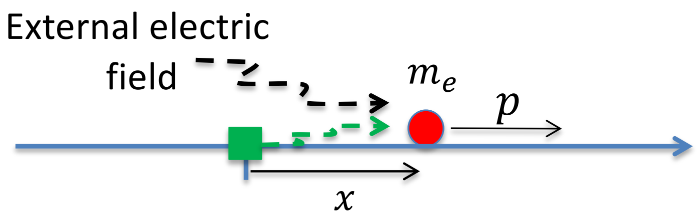
- the particle position assumes values belonging to the interval (−∞,+∞) continuum spectrum
- the particle linear momentum assumes values belonging to the interval (−∞,+∞) continuum spectrum
- the particle energy can assume values belonging to a discrete set $\{E_0, E_1, E_2, \ldots\}$ discrete spectrum

Fundamental Physical Quantities



In this toy problem, the degree of freedom of the system is the position of the particle and the linear momentum is the conjugate momentum: all the other physical quantities can be obtained from them.

Compatible and Incompatible Physical Quantities

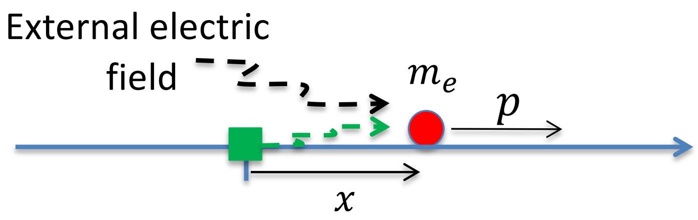


In **classical mechanics** all the physical quantities can, in principle, be measured simultaneously with infinite precision.

Quantum mechanics sets a limit to the precision with which the physical quantities can be measured simultaneously.

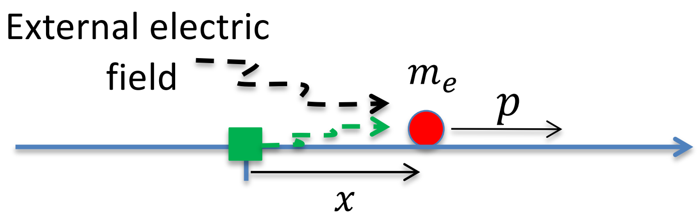
In **Quantum Mechanics** two physical quantities are said to be compatible if they can simultaneously be measured with any precision, otherwise they are said to be incompatible.

Compatible and Incompatible Physical Quantities



- The position x and the conjugated linear momentum p are incompatible according to the Heisenberg's uncertainty relations.
- Canonically conjugate variables are always incompatible, while different degrees of freedom are compatible with one another.
- …

Complete Set of Compatible Physical Quantities



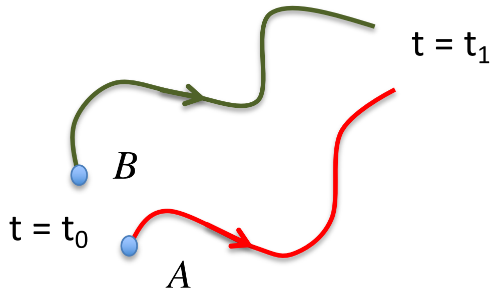
A **<u>complete set of compatible physical variables</u>** is the maximal set of independent and compatible physical quantities of the system.

In the toy problem we have considered, a complete set of compatible physical quantities is composed by <u>only one quantity</u>: the position of the electron <u>or</u> the linear momentum <u>or</u> the energy (we are disregarding the spin degree of freedom).

State

The term **state** has various, more specific meanings in **classical mechanics** and in **quantum mechanics**, but all include the notion that a knowledge of the state is sufficient to make predictions about the future behavior of the system:

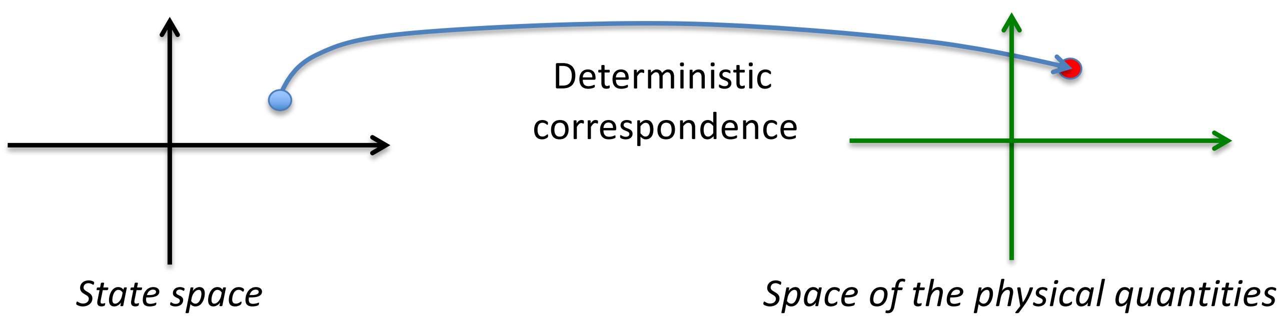
known the state of the system at time t_0 and the laws that govern it, the evolution of the state of the system is uniquely determined for $t > t_0$ (**Principle of Causality**).



We must distinguish between causal and deterministic !!!

Classical Mechanics is causal and deterministic, while Quantum Mechanics is causal, but it is not deterministic.

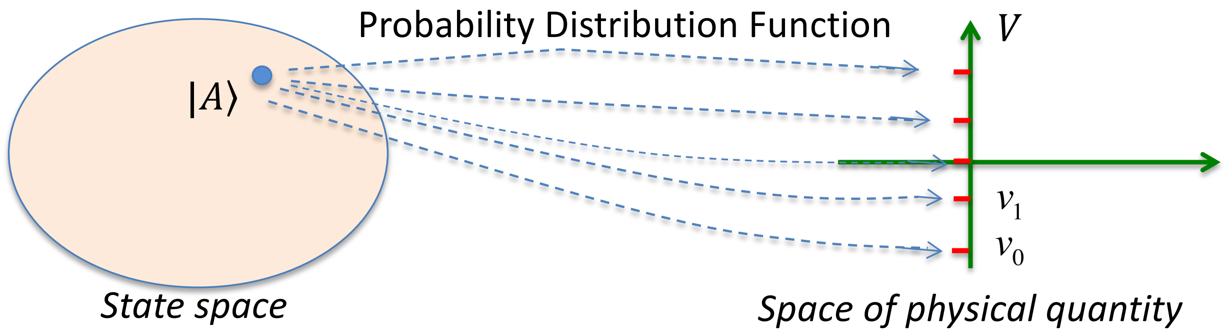
State in Classical Mechanics



In **classical mechanics** the state coincides with the fundamental physical quantities of the particle.

What makes Classical Mechanics deterministic is that the knowledge of the state also determines all possible physical quantities precisely.

State in Quantum Mechanics

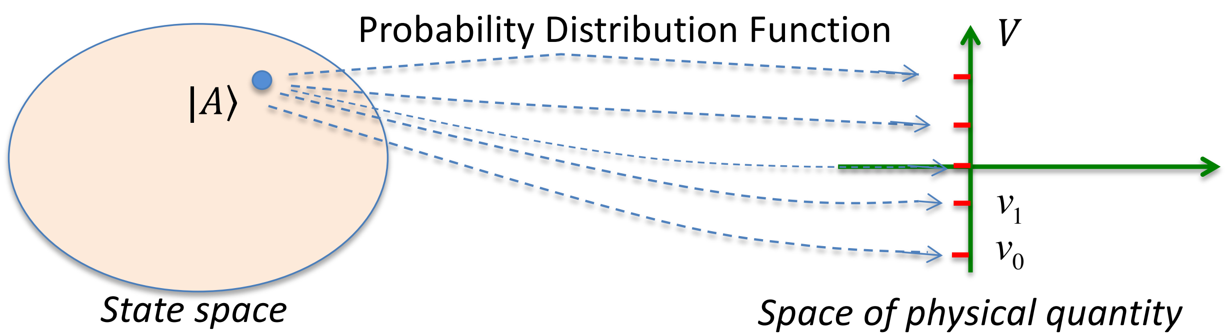


In **quantum mechanics** the relationship between the state and the physical quantities is much less direct:

the state of the particle $|A\rangle$ does not determine the values of the physical quantity V, but only the **probabilities** $P_V(v_0)$, $P_V(v_1)$, $P_V(v_2)$, … of obtaining in a **measurement** the values v_0, v_1, v_2, … (we are considering a physical quantity with discrete spectrum)

$$P_V(v_0) + P_V(v_1) + P_V(v_2) + \cdots = 1$$

State in Quantum Mechanics



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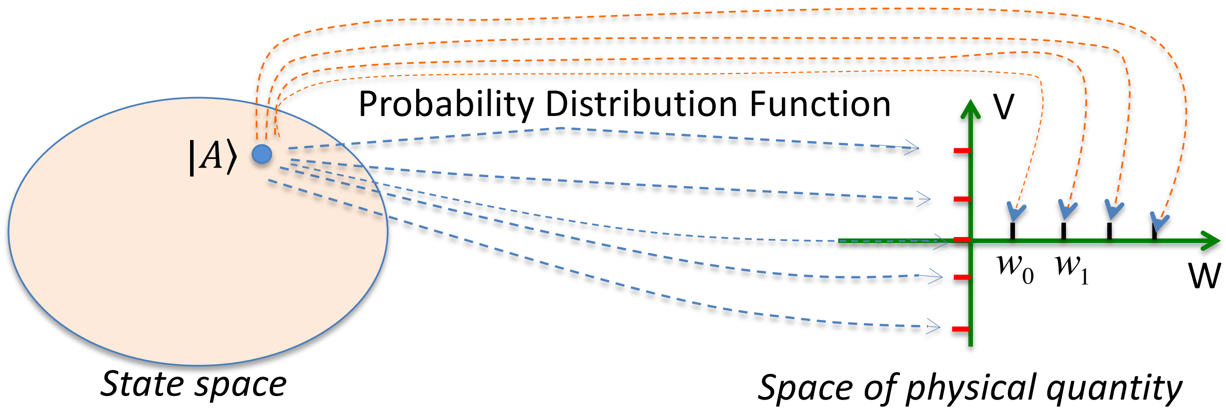
$$P_V(v_0) + P_V(v_1) + P_V(v_2) + \cdots = 1$$

In other words, if you know the state, you can then predict what the statistics of the result of repeated trials of measurement of a particular physical property will be. You will have perfectly determinate statistical predictions but no longer individual predictions.

State in Quantum Mechanics



$P_V(v_n)$ is the probability that the measurement of V when the particle is in the quantum state $|A\rangle$ yields the value v_n,

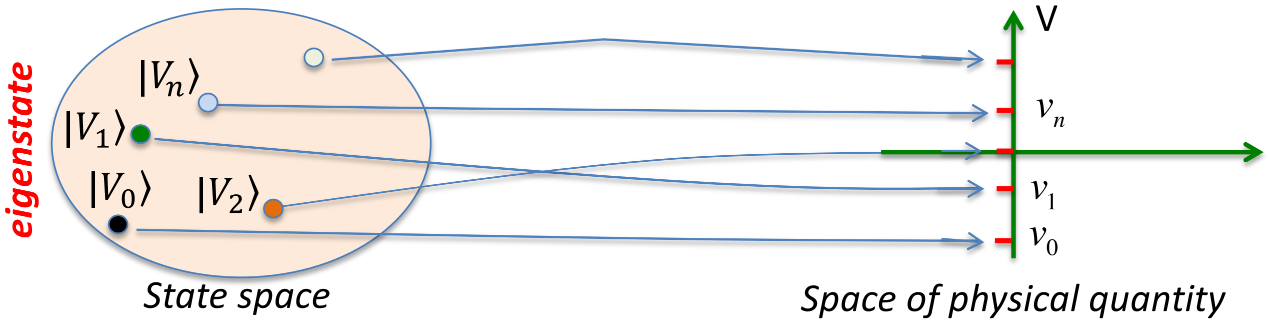
$$P_V(v_0) + P_V(v_1) + P_V(v_2) + \cdots = 1$$

$P_V(w_n)$ is the probability that the measurement of W when the particle is in the quantum state $|A\rangle$ yields the value w_n,

$$P_V(w_0) + P_V(w_1) + P_V(w_2) + \cdots = 1$$

$$\{P_V(v_0),\ P_V(v_1),\ P_V(v_2), \ldots\} \overset{\mathcal{F}}{\Leftrightarrow} \{P_V(w_0),\ P_V(w_1),\ P_V(w_2), \ldots\}$$

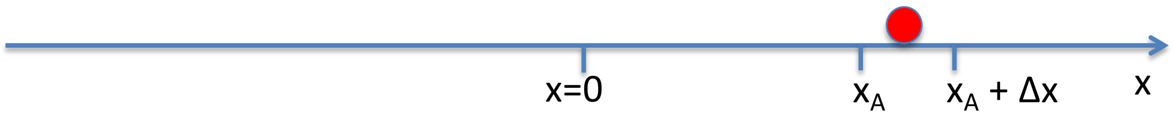
Eigenstates of a Physical Quantity



There exist states in which the measurement of a **complete set of compatible physical quantities** gives certain values: they are the so-called eigenstates of the complete set of compatible physical quantities.

$$|V_n\rangle \rightarrow v_n$$

Superposition of States

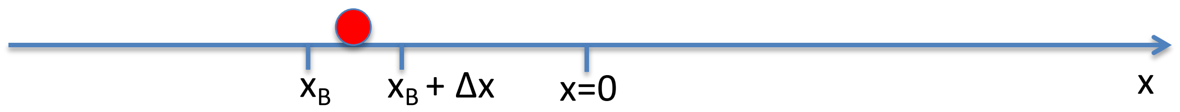


Electron with one Degree of Freedom *(we are disregarding the spin)*

Let us indicate with:

- $|A\rangle$ the eigenstate of the particle in which the measurement of the position gives for certain the result (x_A, $x_A + \Delta x$);

Superposition of States



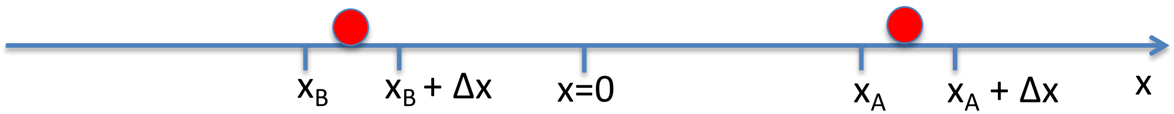
Electron with one Degree of Freedom (we are disregarding spin)

Let us indicate with:

- $|A\rangle$ the eigenstate of the particle in which the measurement of the position gives for certain the result (x_A, $x_A + \Delta x$);

- $|B\rangle$ the eigenstate of the particle in which the measurement of the position gives for certain the result (x_B, $x_B + \Delta x$).

Superposition of States



Electron with one Degree of Freedom (we are disregarding spin)

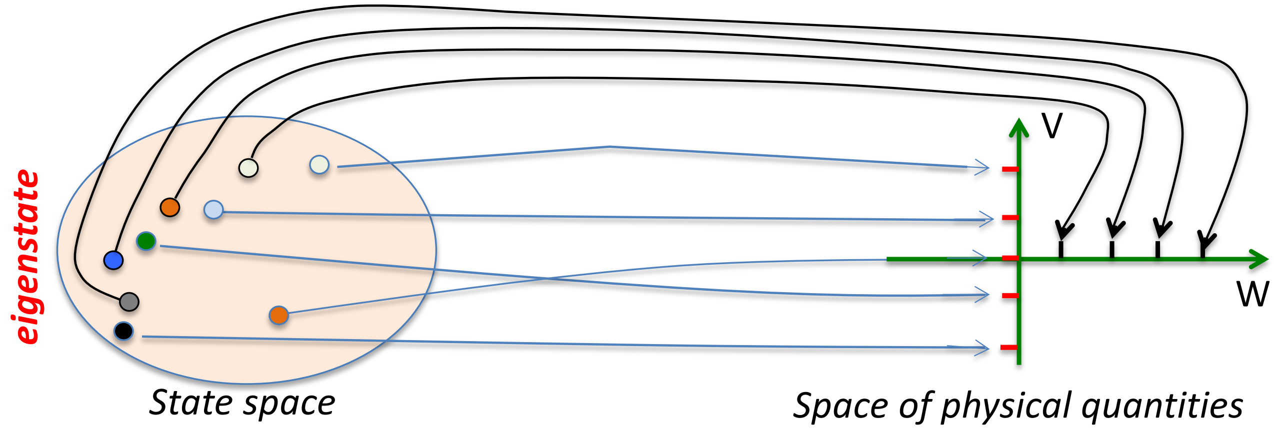
Let us indicate with:

- $|A\rangle$ the eigenstate of the particle in which the measurement of the position gives for certain the result (x_A, $x_A + \Delta x$);
- $|B\rangle$ the eigenstate of the particle in which the measurement of the position gives for certain the result (x_B, $x_B + \Delta x$).

There exist states $|AB\rangle$ in which the measurement of the position gives some time the result (x_A, $x_A + \Delta x$) and sometime the results (x_B, $x_B + \Delta x$) according with a certain probability law:

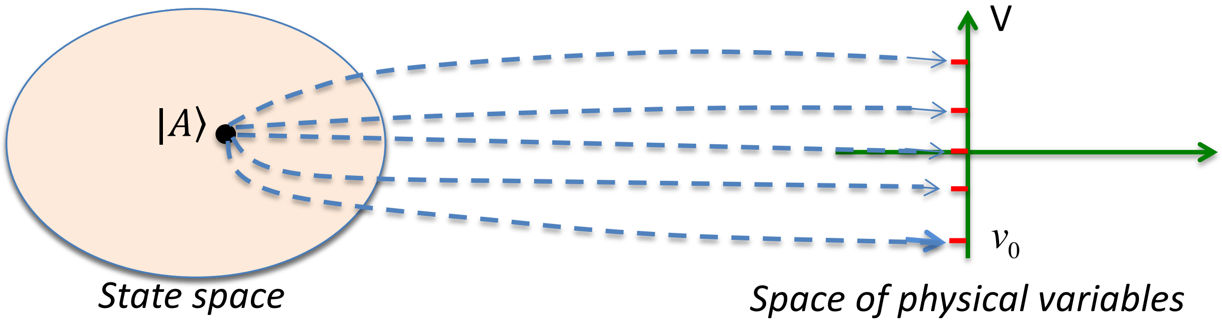
states $|AB\rangle$ are represented through a "linear superposition" of the eigenstates $|A\rangle$ and $|B\rangle$. The relative weights in the superposition are related to the corresponding probabilities.

Incompatible Physical Quantities



Incompatible physical quantities have different sets of eigenstates !!!

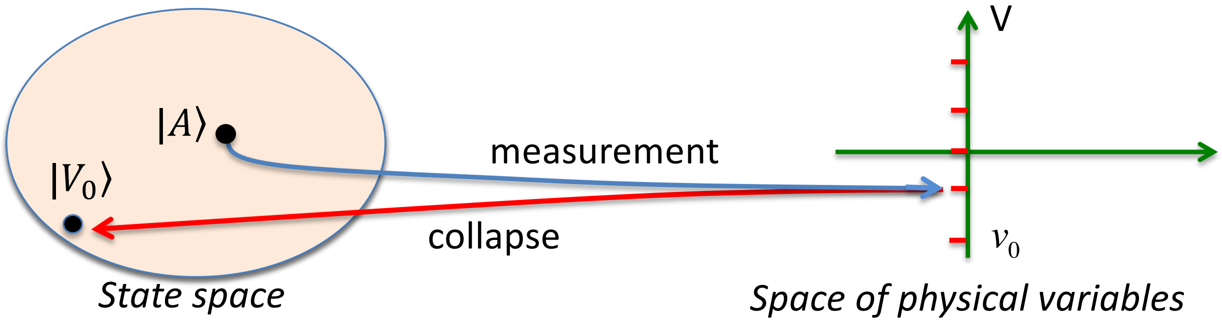
Measurement and Collapse



The system is in the state $|A\rangle$ and we measure the observables *V*.

"*A measurement always causes the system to jump into an eigenstate of the dynamical variable that is being measured, the eigenvalue of this eigenstate belongs to being equal to the result of the measurement*." (P.A.M. Dirac, **The Principles of Quantum Mechanics**)

Measurement and Collapse



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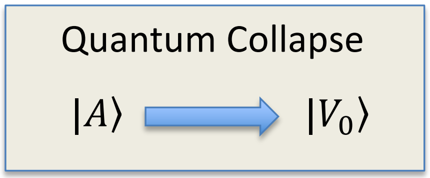
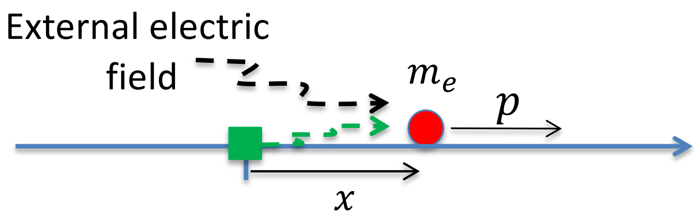


This is an irreversible process !!!

2.1.2 A Look to Quantum Mechanics: Postulates

C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. I, 2nd ed., Wiley-VCH, 2019.

Quantum Mechanic Postulates



We now give a look to the Postulates of Quantum Mechanics by referring to the toy mechanical system (for our purpose, we disregard the intrinsic angular momentum of the electron). The extension to the superconducting circuit with one degree of freedom is immediate.

For the sake of clarity, we use directly the so-called x-representation, that is, we use as basis for the state space of the particle the eigenstates of its position.

First Postulate: State Space

State Space of the System. At any fixed time t, the state of the particle is represented by a square integrable complex function of the position and time $\psi = \psi(x;t)$, the so called wave function, with the constraint $\|\psi\| = 1$. The state space is a Hilbert space, which we denote with S.

$$\|\psi\| = \sqrt{\langle\psi|\psi\rangle} \text{ where } \langle\varphi|\phi\rangle = \int_{-\infty}^{+\infty} dx\, \varphi^*(x)\, \phi(x)$$

norm of the state

scalar product between two states

First Postulate: State Space

State Space of the System. At any fixed time t, the state of the particle is represented by a square integrable complex function of the position and time $\psi = \psi(x; t)$, the wave function, with $\|\psi\|$=1. The state space is a Hilbert space, which we denote with S.

The wavefunction contains all the information relevant to the physical state of the particle.

This postulate implies the **Superposition of States**:
a linear combination of states of the system is another state of the system.

Second Postulate: Observables

Observables. Every measurable physical quantity V is described by a self–adjoint linear operator $\hat{V}$ acting on the wave function, called **observable** and denoted by $\hat{V}$.

$$\hat{V}\varphi(x) = \chi(x)$$

Unlike classical mechanics, quantum physics describes in a fundamentally different manner the state of a system and its physical variables:

a state is represented by a wavefunction belonging to a Hilbert space S, and a physical variable is represented by a self - adjoint linear operator defined in S.

Hermitian Adjoint Operator

The Hermitian adjoint operator $\hat{O}^\dagger$of the linear operator $\hat{O}$ satisfies, by definition, the relation

$$\left\langle \hat{O}^\dagger \varphi \middle| \phi \right\rangle = \left\langle \varphi \middle| \hat{O} \phi \right\rangle,$$

that is,

$$\int_{-\infty}^{+\infty} dx \left[\hat{O}^\dagger \varphi(x)\right]^* \phi(x) = \int_{-\infty}^{+\infty} dx [\varphi(x)]^* \left[\hat{O} \phi(x)\right].$$

The linear operator $\hat{O}$ is self-adjoint if its Hermitian adjoint operator $\hat{O}^\dagger$ acts in the same space S and

$$\hat{O}^\dagger = \hat{O}.$$

Eigenvalues and Eigenfunctions of a Self-Adjoint Operator

eigenvalue

$$\hat{V}\varphi_v(x) = v\varphi_v(x)$$

eigenfunction

The eigenvalues of a linear self-adjoint operator are real and the eigenfunctions with different eigenvalues are orthonormal.

- Discrete spectrum

$$\hat{V}\varphi_n(x) = v_n\varphi_n(x)$$

$v_0,\quad v_1\quad,\quad v_2,\quad \dots$

$\varphi_0(x),\quad \varphi_1(x),\quad \varphi_2(x),\quad \dots$

$$v_m \neq v_n \Rightarrow \langle\varphi_m|\varphi_m\rangle = \delta_{mn}$$

Eigenvalues and Eigenfunctions of a Self-Adjoint Operator

eigenvalue

$$\hat{V}\varphi_v(x) = v\varphi_v(x)$$

eigenfunction

The eigenvalues of a linear self-adjoint operator are real and eigenfunctions with different eigenvalues are orthonormal.

- Continuum spectrum

$$\hat{V}\varphi_v(x) = v\varphi_v(x)$$

$$a \le v \le b$$

$$\downarrow$$

$$\varphi_v(x)$$

$$\langle \varphi_v | \varphi_{v'} \rangle = \delta(v - v')$$

Third Postulate: Values of Physical Quantities and Eigenstates

Values of Physical Quantities and Eigenstates. The only possible results of the measurement of V are the eigenvalues of the corresponding **observable** $\hat{V}$,

$$\hat{V}\varphi_v(x) = v\varphi_v(x),$$

and the eigenfunction $\varphi_v(x)$ associated to the eigenvalue v is the corresponding eigenstate.

When the system is in the state $\varphi_v(x)$ the measurement of V gives certainly the value v.

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When the system is in the state $\varphi_v(x)$ the measurement of V gives certainly the value v.

- To compatible physical quantities correspond commuting observables.
- To incompatible physical quantities correspond non commuting observables.

Correspondence Principle

Quantization rules allow to construct, for any physical quantity V, already defined in classical physics, the observable $\hat{V}$, which describes it in quantum physics.

They are obtained by the **Correspondence Principle:** Quantum Mechanics must reduce to Classical Mechanics when the action is much greater than $\hbar$, that is, as the quantum of action $\hbar$ goes to zero.

Observables

Physical quantity	Classical	Observable (in the x – representation)
Particle Position	x	$\hat{X} = x$
Particle Linear Momentum	p	$\hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial x}$

fundamental

$$[\hat{X}, \hat{P}] \equiv \hat{X}\hat{P} - \hat{P}\hat{X} = i\hbar$$

Fundamental commutation relation

To <u>incompatible</u> physical quantities correspond <u>non commuting observables.</u>

This commutation relation implies the Heisenberg's uncertainty relation and vice versa.

Observables

Physical quantity	Classical	Observable (in the x – representation)
Particle Position	x	$\hat{X} = x$
Particle Linear Momentum	p	$\hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial x}$
Particle Kinetic energy	$K = \frac{1}{2m_e}p^2$	$\hat{K} = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2}$
Particle Potential energy	$W_a = W_a(x)$	$\hat{H} = W_a(x)$
Particle energy	$E = \frac{1}{2m_e}p^2 + W_a(x)$	$\hat{E} = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2} + W_a(x)$
Particle Hamiltonian	$H = \left[\frac{1}{2m_e}p^2 + W_a(x)\right] - F(t)x$	$\hat{H} = \left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2} + W_a(x)\right] - F(t)x$

fundamental (Particle Position, Particle Linear Momentum)

Position Observable

$$\hat{X} = x$$

Eigenvalue problem $$x\varphi_{x_0}(x) = x_0\varphi_{x_0}(x)$$

Continuous spectrum $$-\infty < x_0 < +\infty$$

Eigenstates (in Rigged Hilbert space) $$\varphi_{x_0}(x) = \delta(x - x_0)$$

Linear Momentum Observable

$$\hat{P} = \frac{\hbar}{i}\frac{\partial}{\partial x}$$

Eigenvalue problem $$\frac{\hbar}{i}\frac{\partial}{\partial x}\varphi_p(x) = p\varphi_p(x)$$

Continuous spectrum $$-\infty < p < +\infty$$

Eigenstates (in Rigged Hilbert space) $$\varphi_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$ de Broglie matter wave function

de Broglie wavelength $\lambda_D = 2\pi\frac{\hbar}{p}$

If the characteristic length of the system is of the order of de Broglie wavelength of the particle quantum mechanics effects are important.

Energy Observable

$$\hat{E} = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2} + W_a(x)$$

Eigenvalue problem

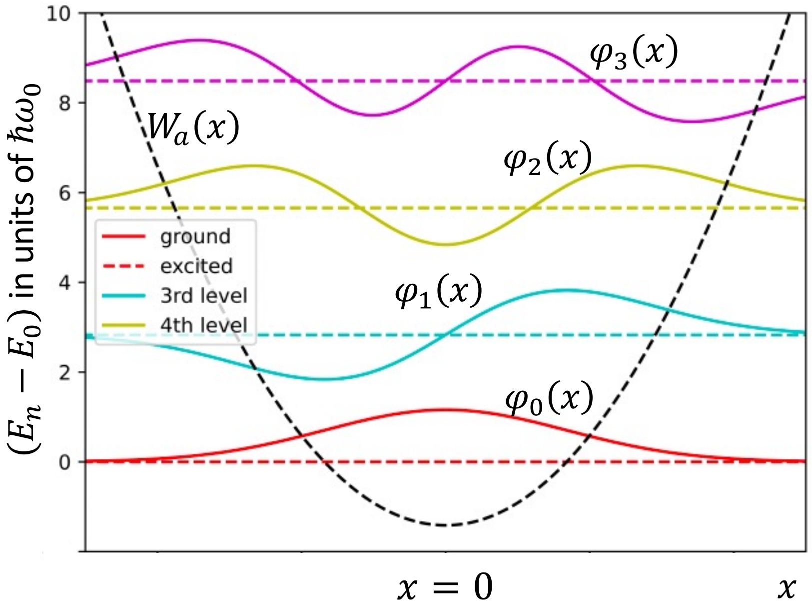
$$\left[-\frac{\hbar^2}{2m_e}\frac{d^2}{dx^2} + W_a(x)\right]\varphi_E(x) = E\varphi_E(x)$$

The spectrum may be discrete and/or continuous, it depends on the potential energy $W_a(x)$.

Energy Observable

Harmonic oscillator $W_a(x) = \frac{1}{2}k_0x^2$

$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right)$ with $n = 0,1,2,\ldots$ and $\omega_0 = \sqrt{k_0/m_e}$



The eigenfunctions can be represented analytically through Hermite polynomials and the Gaussian function.

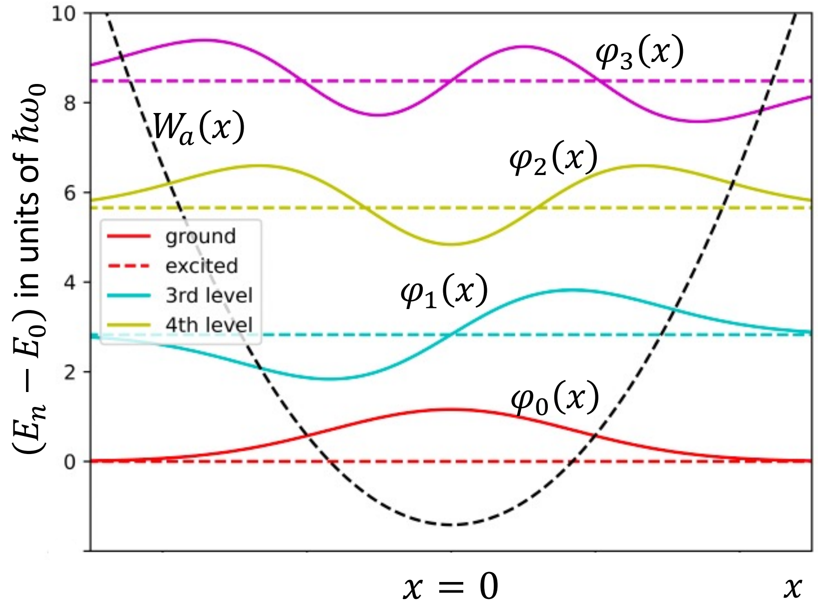
The distance between the energy levels is uniform.

Energy Observable

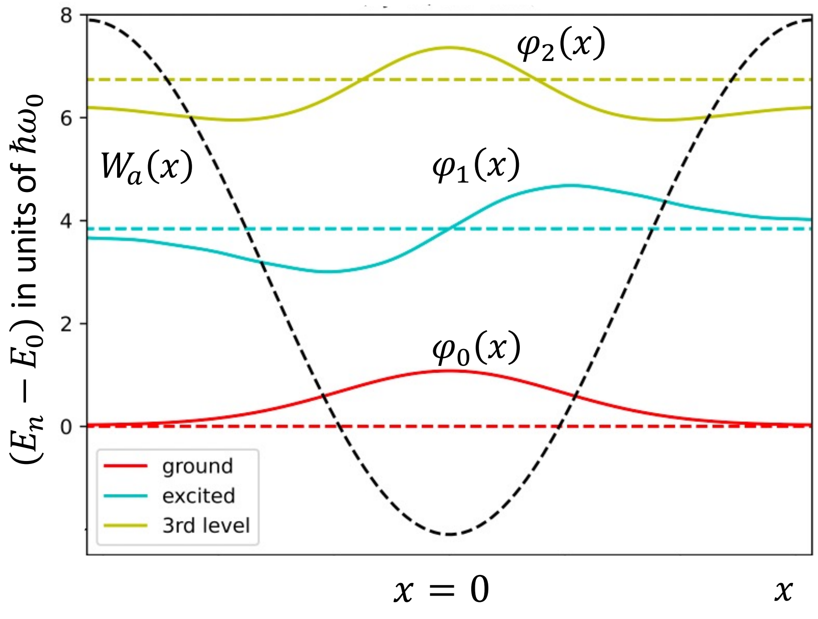
Harmonic oscillator $W_a(x) = \frac{1}{2}k_0x^2$

Anharmonic oscillator $W_a(x) = \frac{1}{2}k_0x^2 + \sigma_4x^4 + \sigma_6x^6 + \cdots$

$E_n = \hbar\omega_0\left(n + \frac{1}{2}\right)$ with $n = 0,1,2, \ldots$ and $\omega_0 = \sqrt{k_0/m_e}$



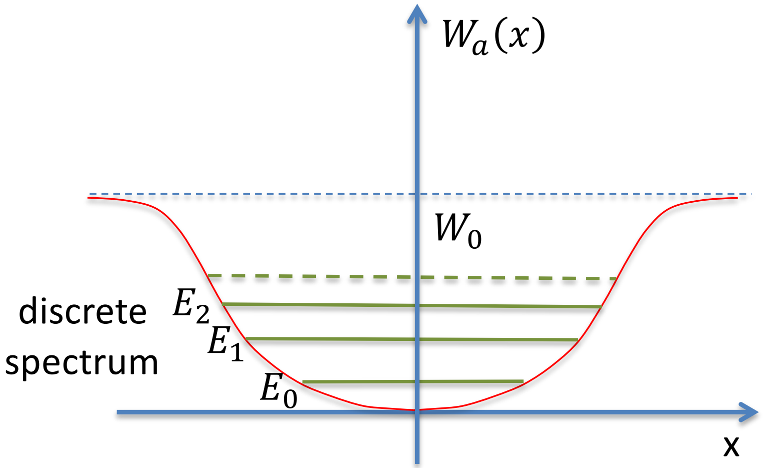
The distance between the energy levels is uniform.



The distance between the energy levels is not uniform.

Energy spectrum: bound and unbound eigenstates

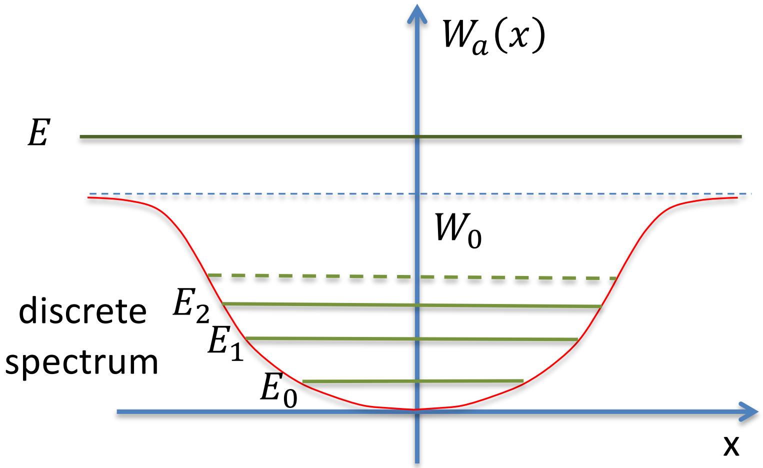
$$\begin{cases} \left[-\frac{\hbar^2}{2m_e}\frac{d^2}{dx^2} + W_a(x)\right]\varphi_E(x) = E\varphi_E(x) \quad -\infty < x < +\infty \\ \text{regularity conditions for } x \rightarrow \pm\infty \end{cases}$$



Bound eigenstates of the particle: are the energy eigenstates $\varphi_0(x)$, $\varphi_1(x)$, …. corresponding to the discrete eigenvalues E_0, E_1, … . They are square-integrable functions, thus $\lim_{x\to\pm\infty} \varphi_n(x) = 0$.

Energy spectrum: bound and unbound eigenstates

$$\begin{cases} \left[-\frac{\hbar^2}{2m_e}\frac{d^2}{dx^2} + W_a(x)\right]\varphi_E(x) = E\varphi_E(x) \quad -\infty < x < +\infty \\ \text{regularity conditions for } x \to \pm\infty \end{cases}$$



$E_a = W_0,\ E_b = \infty$

Unbound eigenstates

$$\lim_{x\to\pm\infty} \varphi_E(x) = \varphi_0(E)e^{\pm ip(E)x/\hbar} \quad E_a < E < E_b$$

$$p = \sqrt{\frac{E}{2m_e}}$$

For $x \to \pm\infty$ the unbounded eigenstates behave as the de Broglie matter waves.

Fourth Postulate: Born rule

Born rule: *Case of a non-degenerate discrete spectrum*. When the physical quantity V is measured with the particle in the normalized state $\psi(x;t)$, the probability $P_V(v_n|\psi)$ of obtaining the value v_n is given by

$$P_V(v_n|\psi) = |\langle\varphi_n|\psi\rangle|^2 = \left|\int_{-\infty}^{+\infty} dx\varphi_n^*(x)\psi(x;t)\right|^2$$

where $\varphi_n(x)$ is the normalized eigenket corresponding to the eigenvalue v_n.

The observable $\hat{V}$ has discrete spectrum $\begin{cases} \hat{V}\varphi_n(x) = v_n\,\varphi_n\,(x) \\ \langle\varphi_m|\varphi_n\rangle = \delta_{mn} \end{cases}$

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Let us represent the wave function $\psi(x;t)$ through the eigenfunctions of the energy observables $\{\phi_n(x)\}$,

$$\psi(x;t) = \sum_n c_n(t)\,\phi_n(x).$$

By using the orthonormality of the energy eigenfunctions we immediately obtain

$$P_E(E_n|\psi) = |c_n(t)|^2$$

where $|c_0(t)|^2 + |c_1(t)|^2 + |c_2(t)|^2 + \ldots = 1$ because $\|\psi\| = 1$.

Fourth Postulate: Born rule

Fourth Postulate: *Case of a non-degenerate continuous spectrum*. When the physical quantity V is measured on a system in the normalized state $\psi(x;t)$ the probability $dP_V(v|\psi)$ of obtaining a result included between v and $v + dv$ is equal to

$$dP_V(v|\psi) = |\langle\varphi_v|\psi\rangle|^2 \mathrm{d}v = \left|\int_{-\infty}^{+\infty} dx \varphi_v^*(x)\psi(x;t)\right|^2 dv$$

where $\varphi_v(x)$ is the normalized eigenket corresponding to the eigenvalue v; $|\langle\varphi_v|\psi\rangle|^2$ is a density of probability.

The observable $\hat{V}$ has continuous spectrum

$$\begin{cases} \hat{A}\, \varphi_v(x) = v|\, \varphi_v(x) \\ \langle\varphi_{v'}|\varphi_n\rangle = \delta(v' - v) \end{cases}$$

Fourth Postulate: Born rule

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where $\varphi_v(x)$ is the normalized eigenket corresponding to the eigenvalue v; $|\langle\varphi_v|\psi\rangle|^2$ is a density of probability.

If V is the position of the particle, we have $\varphi_{x_0}(x) = \delta(x - x_0)$, therefore

$$dP_x(x_0|\psi) = |\psi(x_0;t)|^2 \mathrm{d}x$$

with $\|\psi\|$=1, therefore $|\psi(x_0;t)|^2 \mathrm{d}x$ is the probability of obtaining in the measurement of the particle position x a value between x_0 and $x_0 + dx$.

Fourth Postulate: Born rule

Fourth Postulate: *Case of a non-degenerate continuous spectrum*. When the physical quantity V is measured on a system in the normalized state $\psi(x;t)$ the probability $dP_V(v|\psi)$ of obtaining a result included between v and $v+dv$ is equal to

$$dP_V(v|\psi) = |\langle\varphi_v|\psi\rangle|^2 \mathrm{d}v = \left|\int_{-\infty}^{+\infty} dx \varphi_v^*(x)\psi(x;t)\right|^2 dv$$

where $\varphi_v(x)$ is the normalized eigenket corresponding to the eigenvalue v; $|\langle\varphi_v|\psi\rangle|^2$ is a density of probability.

If V is the linear momentum of the particle, we have $\varphi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$ therefore

$$dP_p(p|\psi) = |\Pi(p;t)|^2 \mathrm{d}p$$

where $\Pi(p;t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} dx e^{-ipx/\hbar} \psi(x;t)$ (it is the Fourier transform of the wavefunction $\psi(x;t)$) and $\int_{-\infty}^{+\infty} dp \ |\Pi(p;t)|^2 = 1$.

Expectation Value of Observables with Discrete Spectrum

Born rule: *Case of a non-degenerate discrete spectrum*. When the physical quantity V is measured with the particle in the normalized state $\psi(x;t)$, the probability $P_V(v_n|\psi)$ of obtaining the value v_n is given by

$$P_V(v_n|\psi) = |\langle\varphi_n|\psi\rangle|^2 = \left|\int_{-\infty}^{+\infty} dx\varphi_n^*(x)\psi(x;t)\right|^2$$

where $\varphi_n(x)$ is the normalized eigenfunction corresponding to the eigenvalue v_n.

Expectation value of V (probabilistic expected value of the measurement) with the particle in the state $\psi(x;t)$

$$\langle V\rangle_\psi = \sum_n P_V(v_n|\psi)\, v_n = \sum_n |\langle\varphi_n|\psi\rangle|^2\, v_n = \sum_n \langle\psi|\,\varphi_n\rangle\, v_n\langle\,\varphi_n|\psi\rangle = \langle\boldsymbol{\psi}|\hat{\boldsymbol{V}}\boldsymbol{\psi}\rangle$$

Statistics

Expectation value of V with the particle in the state $\psi(x;t)$

$$\langle V \rangle_\psi = \langle \psi | \hat{V} \psi \rangle$$

Standard deviation of V with the particle in the state $\psi(x;t)$

$$\Delta V_\psi = \sqrt{\langle V^2 \rangle_\psi - \langle V \rangle_\psi^2}$$

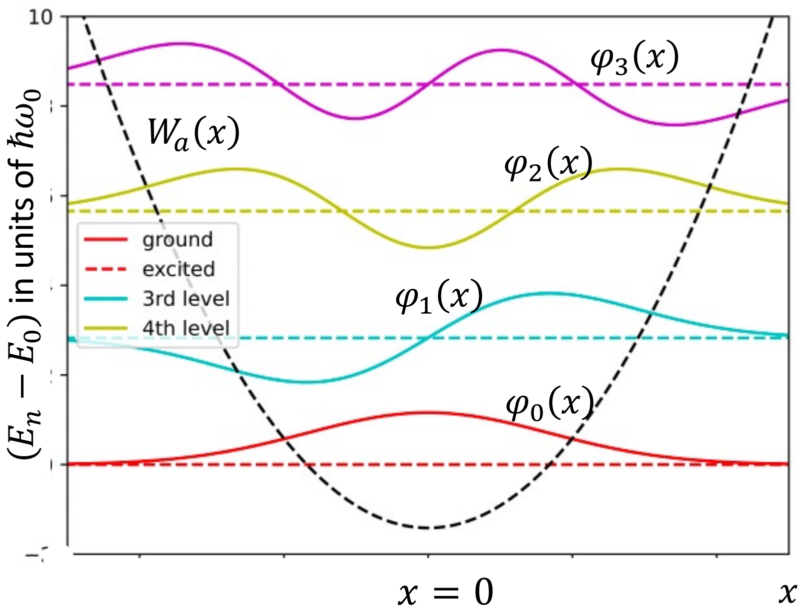
Heisenberg's uncertainty relation

$$\Delta x_\psi \, \Delta p_\psi \geq \frac{\hbar}{2}$$

It is a direct consequence of the commutation relation $\hat{X}\hat{P} - \hat{P}\hat{X} = i\hbar$ and vice versa.

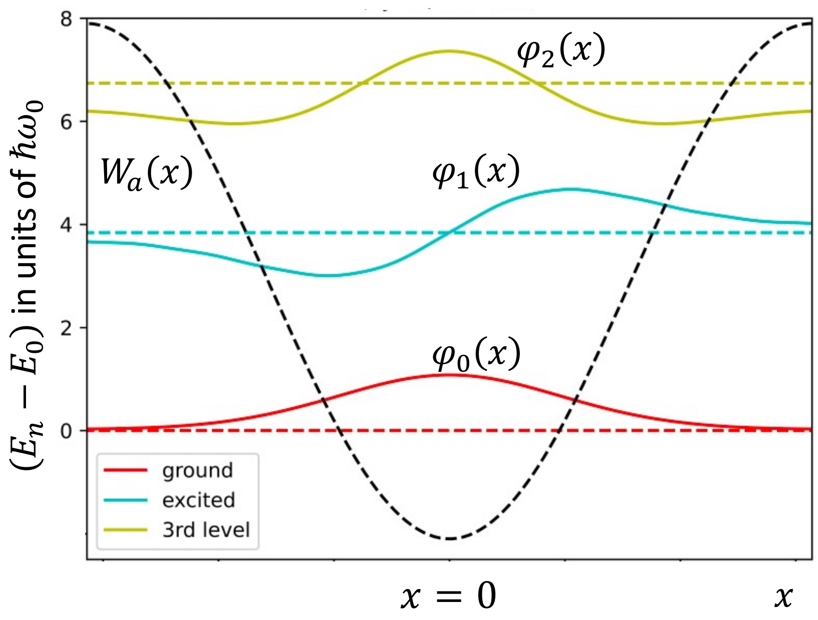
Statistics of Energy Eigenstates

Harmonic oscillator $W_a(x) = \frac{1}{2}k_0x^2$



- $\langle x \rangle_{\varphi_n} = 0, \langle p \rangle_{\varphi_n} = 0;$
- $\Delta x_{\varphi_n} = \sqrt{n + 1/2}x_c, x_c = \sqrt{\hbar/m_e\omega_0}$
- $\Delta p_{\varphi_n} = \sqrt{n + 1/2}p_c, p_c = x_c\, m_e\omega_0$
- $\Delta x_{\varphi_n}\, \Delta p_{\varphi_n} = (n + 1/2)\hbar$

Anharmonic oscillator $W_a(x) = \frac{1}{2}k_0x^2 + \sigma_4x^4 + \sigma_6x^6 + \cdots$

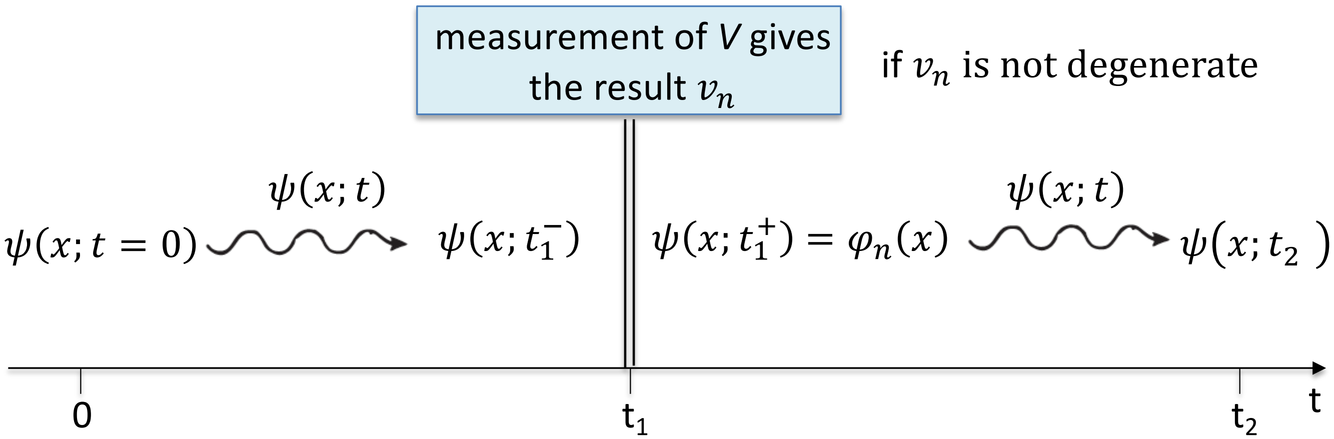


- $\langle x \rangle_{\varphi_n} = 0, \langle p \rangle_{\varphi_n} = 0$ for symmetric potential energies.
- Δx_{φ_n} depends on $W_a(x)$;
- Δp_{φ_n} depends on $W_a(x)$;
- $\Delta x_{\varphi_n}\, \Delta p_{\varphi_n} \geq \hbar/2$

Fifth Postulate: Random Wave Function Collapse

Wave function collapse: *Case of non – degenerate discrete spectrum*. If the measurement of the physical quantity *V* with the particle in the state $\psi(x;t)$ gives the result v_n the state of the particle immediately after the measurement is the eigenstate $\varphi_n(x)$ associated with the eigenvalue v_n ,

$$\psi(x;t) \overset{v_n}{\Rightarrow} \varphi_n(x).$$



Sixth Postulate: Time Evolution of the System

Schrödinger equation. The time evolution of the state $\psi(x;t)$ is governed by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}(t)\,\psi$$

where $\hat{H}(t)$ is the Hamiltonian observable,

$$\hat{H}(t) = \left[-\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2} + W_a(x)\right] - F(t)x,$$

that is,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m_e}\frac{\partial^2\psi}{\partial x^2} + W_a(x)\,\psi - F(t)x\,\psi.$$

The Schrödinger equation is solved with given initial and boundary conditions for the wavefunction $\psi(x;t)$.

According to the **Principle of Causality**

Seventh Postulate: **Identical Particles**

Seventh Postulate: When a system includes several identical particles, only certain wavefunctions can describe its physical states: wavefunctions are either completely symmetric or completely antisymmetric with respect to permutation of the position of the particles, depending on the nature of the identical particles.

Seventh Postulate: **Identical Particles**

Seventh Postulate: When a system includes several identical particles, only certain wavefunctions can describe its physical states: wavefunctions are either completely symmetric or completely antisymmetric with respect to permutation of the position of the particles, depending on the nature of the identical particles.

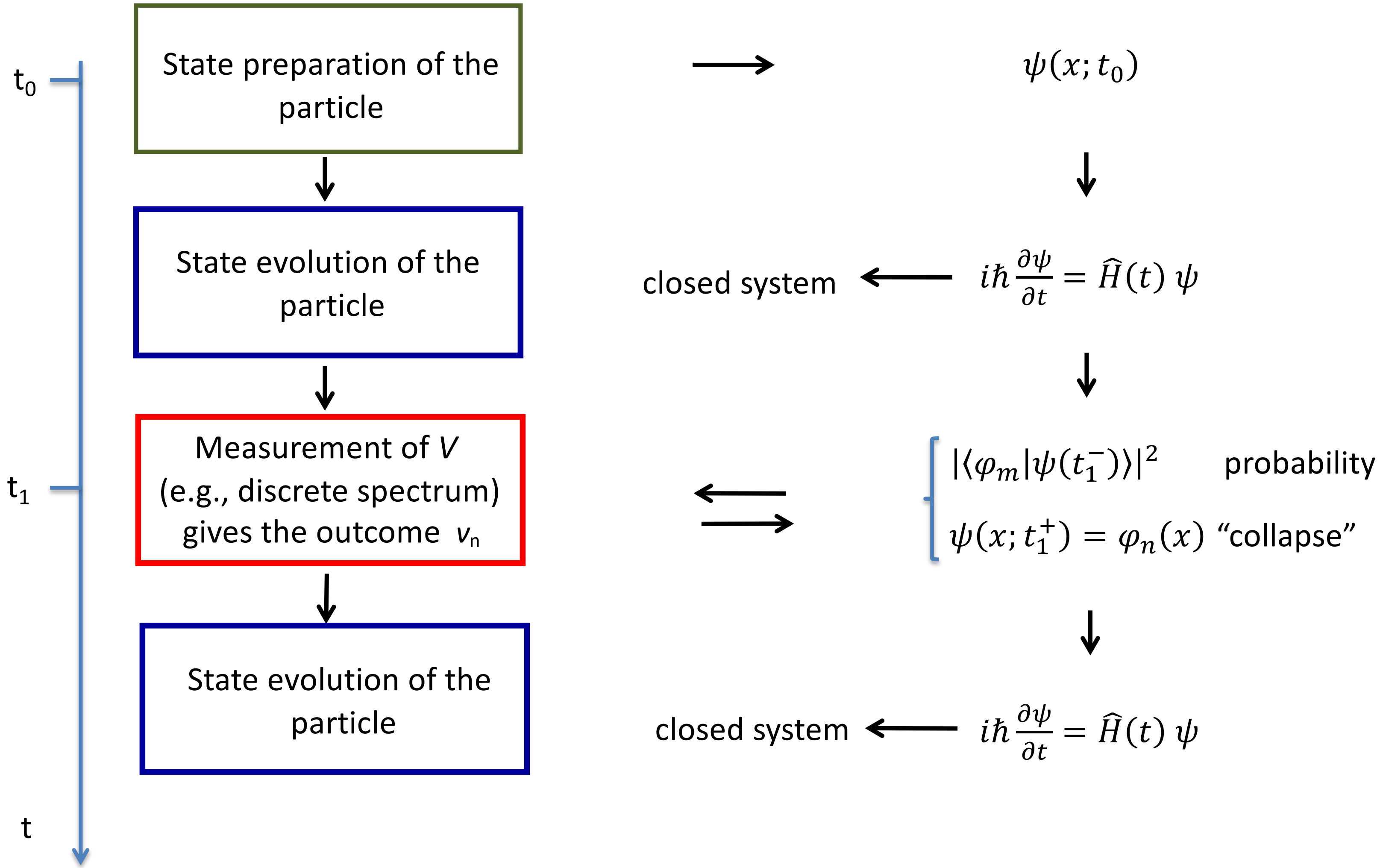
Those particles for which the wavefunctions are symmetric are called **bosons**, and those for which the wavefunctions are antisymmetric, **fermions**.

Electrons are fermions, Cooper pairs in low temperature superconductors behave as boson.

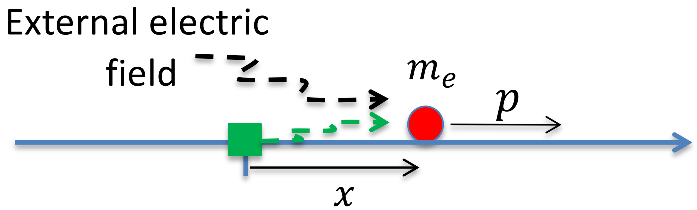
Fermions obey the Pauli exclusion principle: two or more identical fermion particles cannot simultaneously occupy the same quantum state.

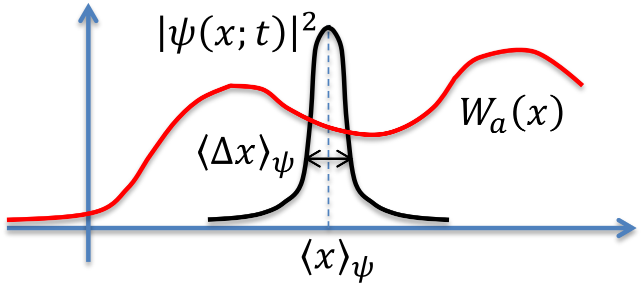
Bosons: two or more identical boson particles can simultaneously occupy the same quantum state.

Schrödinger picture



Ehrenfest's theorem





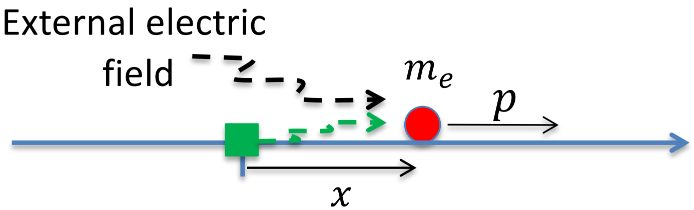
Let us assume $\psi(x;t)$ to be highly localized: $|\psi(x;t)|^2$ takes on non-negligible values only within an interval whose length is much smaller than the distances over which the potential energy $W_a(x)$ varies appreciably.

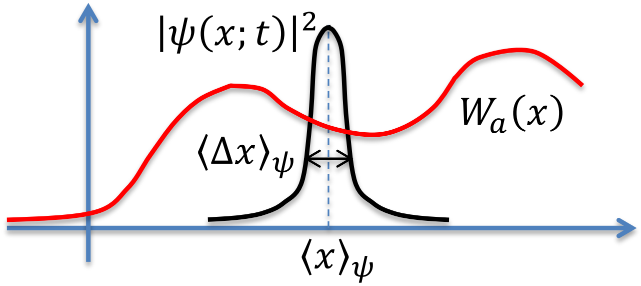
Then the expectation value of the position $\langle x \rangle_\psi$ and the expectation value of the linear momentum $\langle p \rangle_\psi$ are approximately governed by the classical equations of motion,

$$\frac{d}{dt}\langle x \rangle_\psi = \frac{1}{m_e}\langle p \rangle_\psi,$$

$$\frac{d}{dt}\langle p \rangle_\psi \cong -\frac{d}{dx}W_a\left(\langle x \rangle_\psi\right) + F(t).$$

Ehrenfest's theorem





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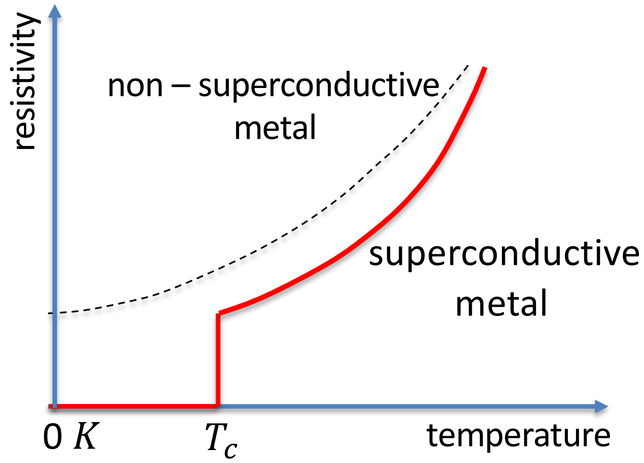
$$\frac{d}{dt}\langle p \rangle_\psi \cong -\frac{d}{dx}W_a\big(\langle x \rangle_\psi\big) + F(t).$$

In the macroscopic limit, the characteristic de Broglie wavelength of the particle is much smaller than the distances over which the potential energy varies, and the wave packets are sufficiently short in space. This result is very important because it shows that the equations of classical mechanics follow from the Schrödinger equation in certain limiting conditions, which are verified for macroscopic objects.

2.2 Superconducting Quantum Circuits

Superconductive metals

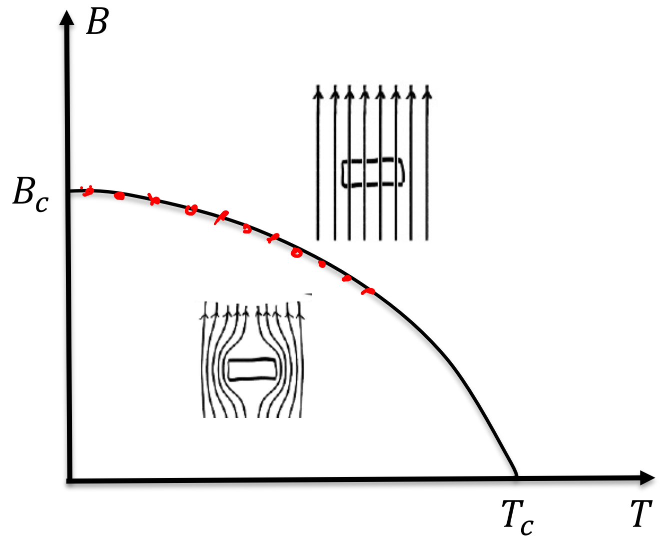


Various metals become **superconducting** below a certain temperature T_c (**transition temperature**), which depends on the material: the resistivity drops to zero when the temperature of the sample is lowered below T_c .

The term "conventional superconductors" refers to those materials with $T_c < 25K$.

T. Orlando, K. A. Delin, Foundations of Applied Superconductivity, Addison Wesley (1991).

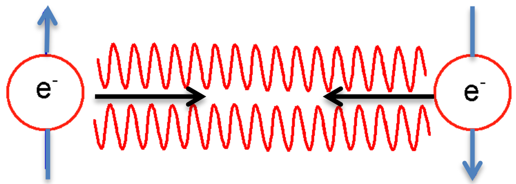
Type - I superconductors



In **Type - I superconductors**, superconductivity is abruptly destroyed when the strength of the applied magnetic field rises above a critical value B_c (critical magnetic field).

Cooper Pairs

The free electrons (normal electrons) in the superconducting state pair together and form the Cooper pairs (super - electrons). The electrons in a Cooper pair are bound with an energy (energy gap 2Δ) that is typically of the order of $10^{-4} \div 10^{-3}$ eV for conventional superconductivity.



The electrons of a Cooper pair have opposite intrinsic angular momenta (spin), thus Cooper pairs behave as boson particles.

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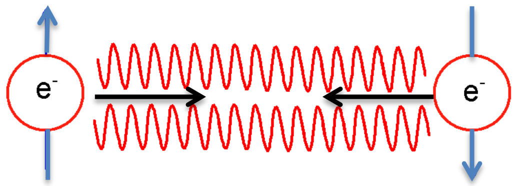


The electrons of a Cooper pair have opposite intrinsic angular momenta (spin), thus Cooper pairs behave as boson particles.

It must be supplied at least an amount of energy equal to 2Δ to split the Cooper pair into two unbound normal electrons and to destroy superconductivity.

Take, for instance, a superconductor maintained at a temperature considerably below T_c, ensuring that all electrons are paired. If we expose this material to electromagnetic radiation, the superconducting properties should remain unchanged unless the radiation energy matches or exceeds the energy gap 2Δ. For conventional superconductors, this energy gap correlates with frequencies ranging from 100 to 1000 GHz.

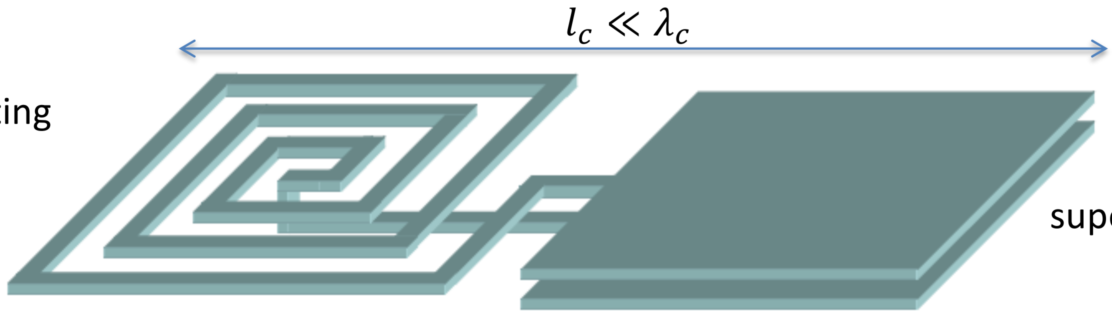
Type – I superconducting metals

Material	T_c (K)	$\Delta(meV)$	B_c(mT)
Al	1.18	0.18	10.5
In	3.41	0.54	23.0
Sn	3.72	0.59	30.5
Pb	7.20	1.35	80.0
Nb	9.25	1.50	198.0

$1 \text{ eV} = 1.602176634\times10^{-19}$ J

Source: T. Orlando, K. A. Delin, Foundations of Applied Superconductivity, Addison Wesley (1991).

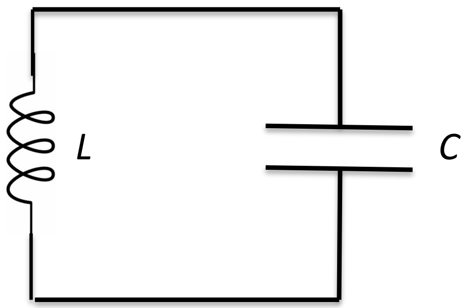
Superconducting LC Circuit



This system contains an enormous number of superelectrons.

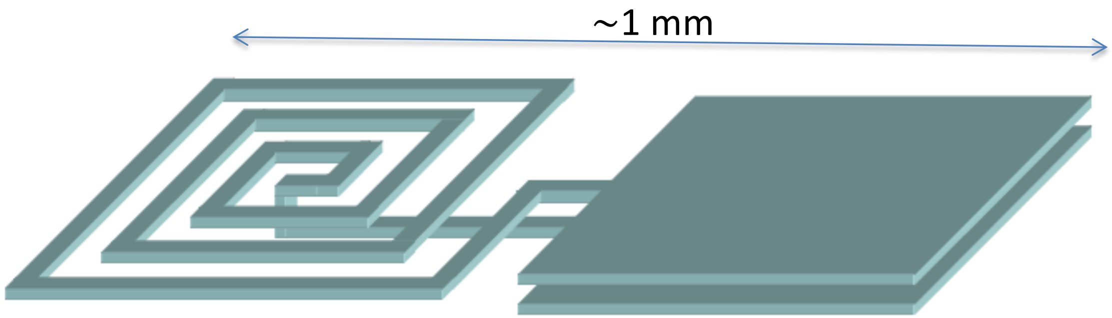
- Superelectrons are bosons, they share the same quantum state.
- The characteristic wavelenght λ_c of the electromagnetic field significantly exceeds the characteristic linear size of the system l_c, thus the system behaves as it would have only one degree of freedom.

Under the above conditions the system may behave as a quantum superconducting circuit.



The degree of freedom of the circuit is the magnetic flux linked with the inductor winding (or the electric charge stored on the capacitor electrodes).

Superconducting LC Circuit



L ≈ 1 nH, C ≈ 10 pF

Natural frequency of the LC circuit $\frac{\omega_0}{2\pi} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}} = 1.69$ ... GHz $\Leftrightarrow \lambda = 18.85$ ... cm

Characteristic impedance $Z_0 = \sqrt{\frac{L}{C}} = 10\ \Omega$

The characteristic impedance is very important because the coupling of the LC circuit with the surrounding environment depends on it.

Observation of macroscopic quantum phenomena in the LC circuit

Two criteria must be satisfied:

i. the "thermal energy" of the circuit $k_B T_0$ must be small compared with the separation of the quantized energy levels $\hbar\omega_0$, $\hbar\omega_0 \gg k_B T_0$;

J. Legget, *Macroscopic quantum systems and the quantum theory of measurement*, Progress of Theoretical Physics Supplement, 1980.

J Clarke, AN Cleland, MH Devoret, D Esteve, Quantum mechanics of a macroscopic variable: the phase difference of a Josephson junction, Science, 239, 1988.

Observation of macroscopic quantum phenomena in the LC circuit

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i. the “thermal energy” of the circuit $k_B T_0$ must be small compared with the separation of the quantized energy levels $\hbar\omega_0$, $\hbar\omega_0 \gg k_B T_0$;

ii. the macroscopic degree of freedom of the circuit must be sufficiently decoupled from the “environment” if the **lifetime of the quantum state must** be enough high.

J. Legget, *Macroscopic quantum systems and the quantum theory of measurement*, Progress of Theoretical Physics Supplement, 1980.
J Clarke, AN Cleland, MH Devoret, D Esteve, Quantum mechanics of a macroscopic variable: the phase difference of a Josephson junction, Science, 239, 1988.

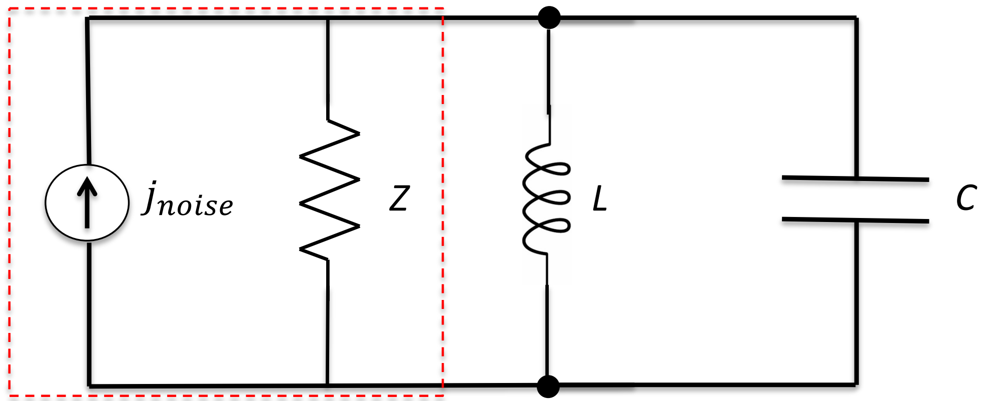
Observation of macroscopic quantum phenomena in the LC circuit

The circuit is "observed" by means of cables connecting room temperature measurement apparatus to base temperature circuit: thermal photons propagate down the cable towards the lower temperature stage.

Norton equivalent one port of the environment:

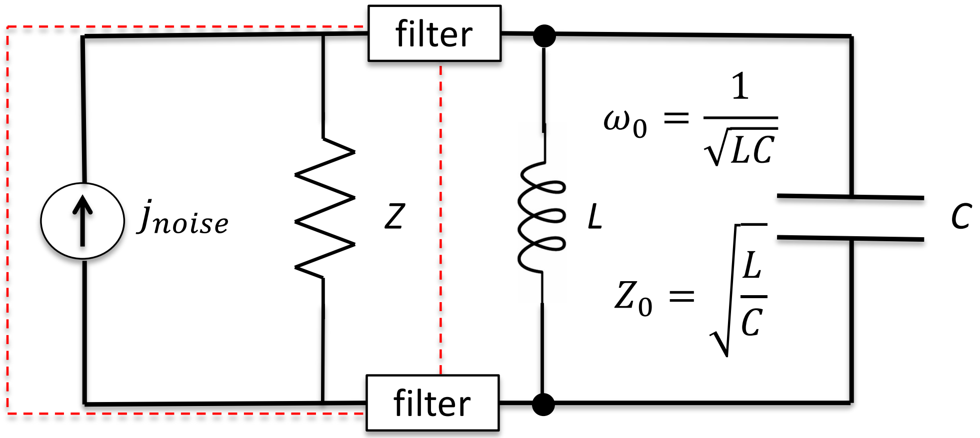
Z is the characteristic impedance of the cable and j_{noise} accounts for thermal photons incoming from measurement apparatus.

The impedance Z induces "relaxation" and the noise current induces "decoherence".



U. Vool, M. Devoret, Introduction to quantum electromagnetic circuits, Int. J. Circ. Theor. Appl. 2017.

Observation of macroscopic quantum phenomena in the LC circuit



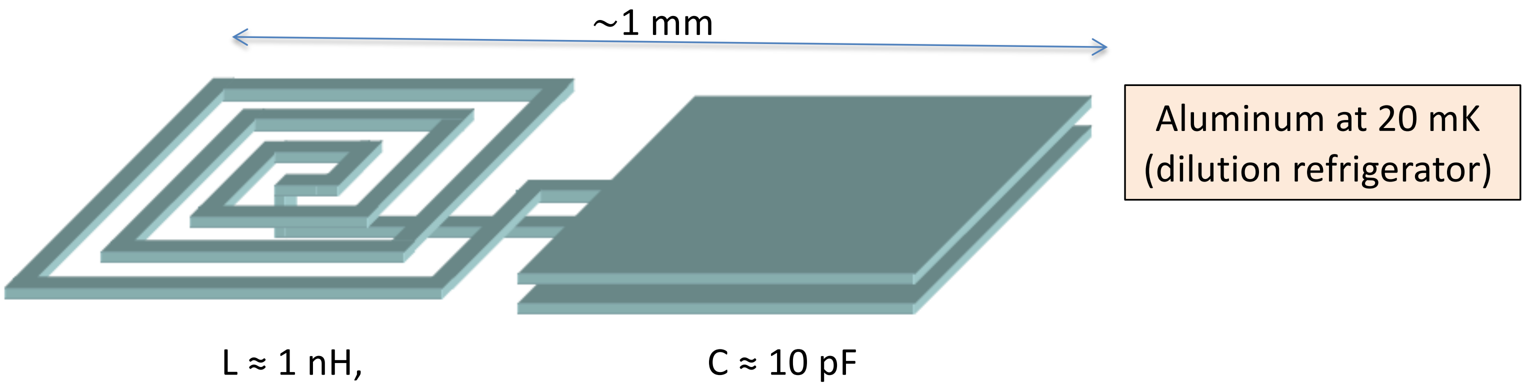
To observe quantum effects, it is required that:

i. $\hbar\omega_0 \gg k_B T_0$: this assures that quantum phenomena are not masked by thermal noise;

ii. the power intensity of thermal radiation incoming from the measurement apparatus must be strongly reduced through a series of filters to levels such that the noise photon number is much smaller than one;

iii. $Z \gg Z_0$: the relaxation resulting from interactions with the environment happens on a much longer time scale compared to the time scale characterizing quantum processes.

Observation of macroscopic quantum phenomena in the LC circuit



- Aluminum has the transition temperature of 1.1 K.
- Dilution refrigerator at 10 mK assures that thermal noise does not mask quantum phenomena
- Aluminum has a gap $2\Delta \cong$ 0.36 m eV. Dissipation due to the breaking of Cooper pairs will begin at frequencies greater than $(2\Delta)/h \cong$ 100GHz.
- the power intensity of thermal radiation incoming from the measurement apparatus is strongly reduced through a series of filters;
- $Z \gg Z_0$: the relaxation resulting from interactions with the environment happens on a much longer time scale compared to the time scale characterizing quantum processes.

Cabled Dilution Refrigerator (cryostat)

Superconducting circuits operate at temperature of 10 ÷ 20 mK, in the frequency range 1 ÷ 20 GHz (microwaves).

The circuits are designed in such a wave to minimize the effects due to the interaction with the environment.

Under these conditions the quantum phenomena are not masked on sufficiently long time intervals.

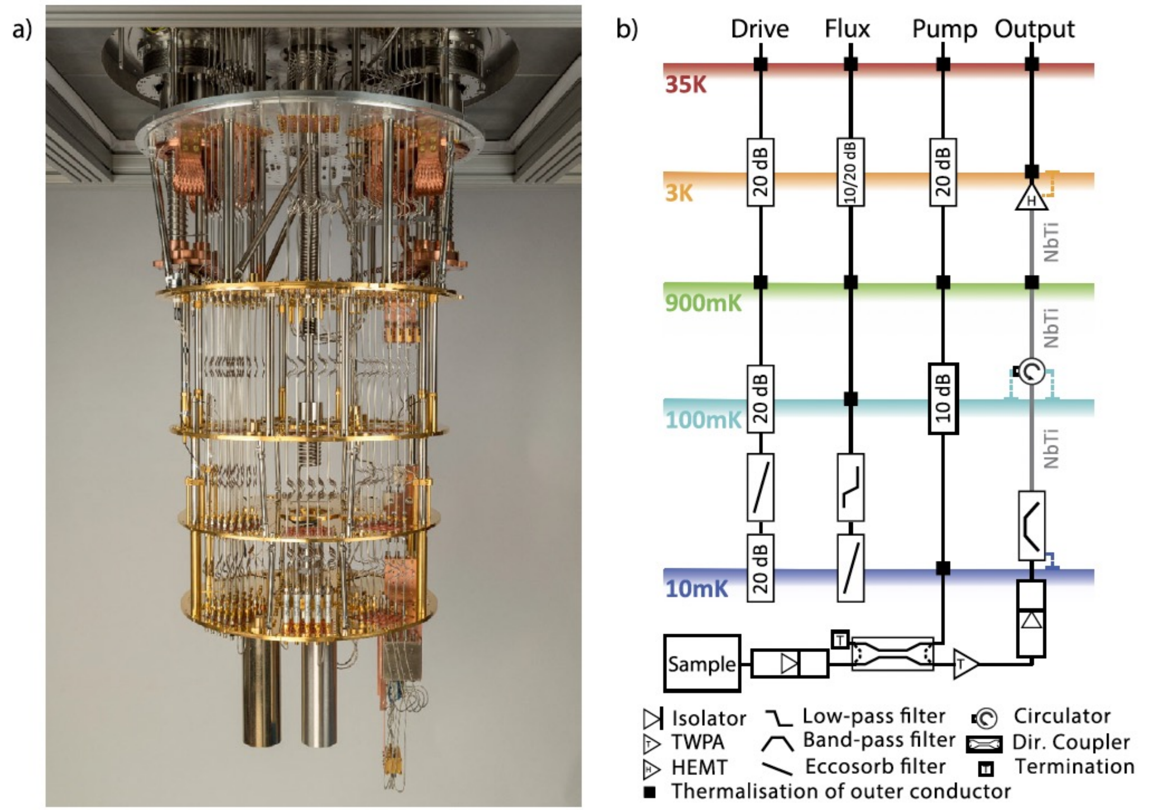
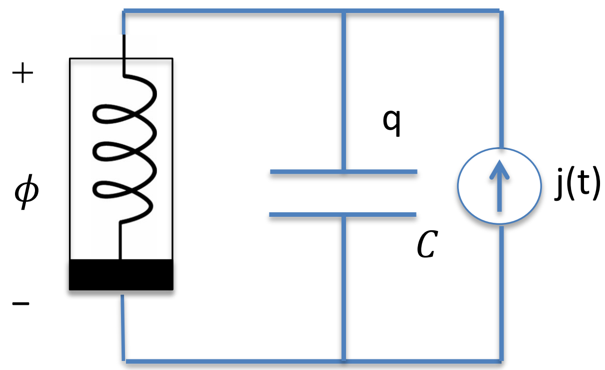


Figure 3 Cabled dilution refrigerator (DR). (**a**) Bluefors XLD DR with 25 drive lines, 25 flux lines, 4 read-out, 6 read-in, and 5 pump lines installed (see end of Sect. 3.1 for details). (**b**) Schematic of the cabling configuration within the DR

S. Krinner *et al.,* Engineering cryogenic setups for 100 qubit scale superconducting circuit systems, EPJ Quantum Technology, 2019.

Quantization of Electrical Circuits

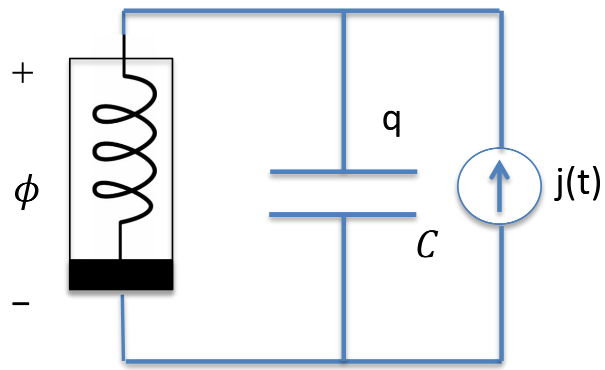


The flux ϕ and the charge q are canonically conjugated variables of this superconducting circuit.

The Hamiltonian of the circuit is:

$$H(q,\phi;t) = \frac{1}{2C}q^2 + W_i(\phi) - j(t)\,\phi.$$

Quantization of Electrical Circuits



Under the aforementioned conditions, the passage from the classical to the quantum description is direct: the canonically conjugated classical variables are replaced by the corresponding observables (operators),

$$q \rightarrow \hat{q},$$
$$\phi \rightarrow \hat{\phi},$$

and the Hamiltonian function is replaced by :

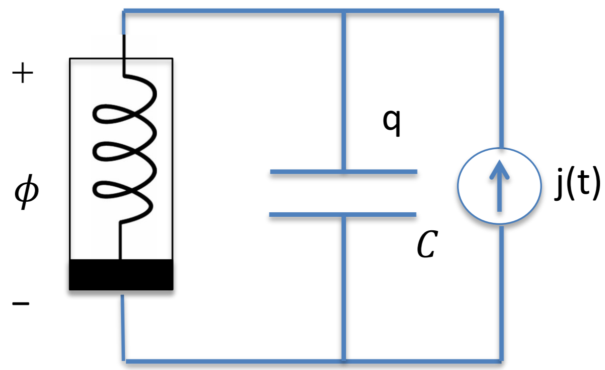
$$H\ (\phi, q; t) \rightarrow \widehat{H} = H\big(\hat{\phi}, \hat{q}; t\big).$$

U. Vool, M. Devoret, Introduction to quantum electromagnetic circuits, Int. J. Circ. Theor. Appl. 2017.

S.E. Rasmussen et al., **Superconducting Circuit Companion—an Introduction with Worked Examples,** PRX Quantum **2**, 040204, **2021.**

A. Ciani, D. P. DiVincenzo, B. M. Terhal, **Lecture Notes on Quantum Electrical Circuits, 2024,** arXiv:2312.05329.

Quantization of Electrical Circuits

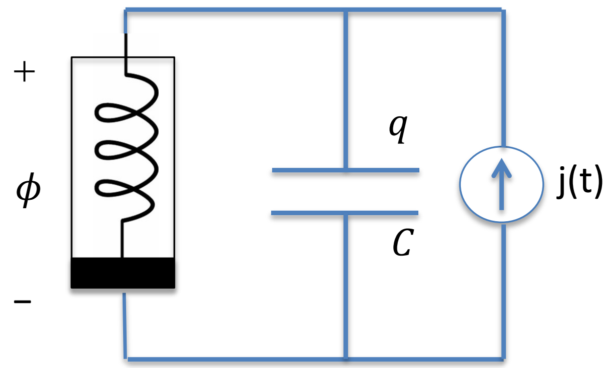


$$q \to \hat{q},$$
$$\phi \to \hat{\phi},$$
$$H(\phi, q; t) \to \widehat{H} = H\big(\hat{\phi}, \hat{q}; t\big).$$

The state of the circuit is represented (in the ϕ −representation) by the wavefunction

$$\Psi = \Psi(\phi; t).$$

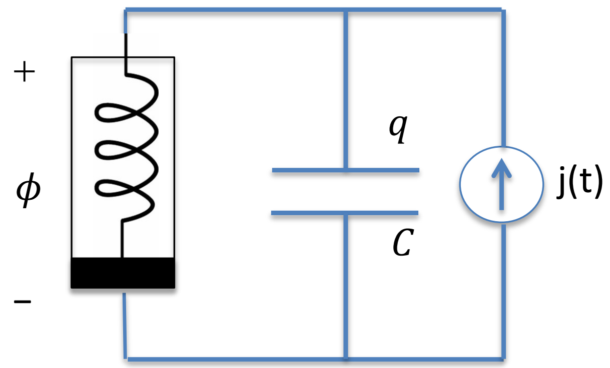
Circuit wavefunctions



Wave function in the flux – representation $\Psi = \Psi(\phi; t)$:

$|\Psi(\phi; t)|^2 d\phi$ is the probability that a measurement of ϕ gives at time t a value belonging to the interval $\phi, \phi + d\phi$.

Circuit wavefunctions



Wave function in the flux – representation $\Psi = \Psi(\phi; t)$:

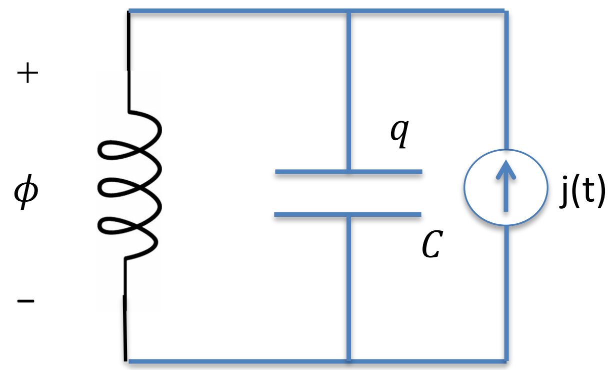
$|\Psi(\phi;t)|^2 d\phi$ is the probability that a measurement of ϕ gives at time t a value belonging to the interval $\phi, \phi + d\phi$.

Wave function in the charge – representation $\Sigma = \Sigma(q; t)$:

$|\Sigma(q;t)|^2 dq$ is the probability that a measurement of q gives at time t a value belonging to the interval $q, q + dq$.

$$\Sigma(q;t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} d\phi e^{-iq\phi/\hbar} \Psi(\phi;t)$$

Linear LC Circuit



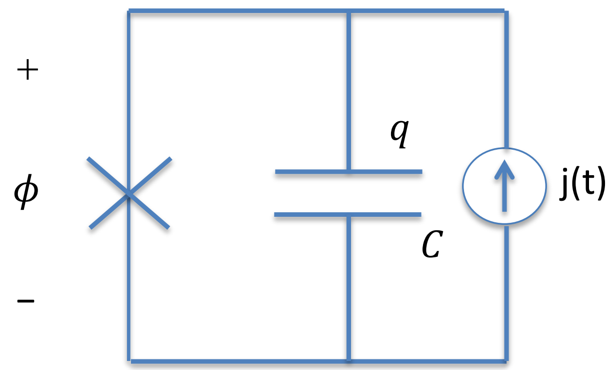
For the linear *LC* circuit, the commutation relation

$$[\hat{\phi}, \hat{q}] = i\hbar$$

can be derived from the quantization of the electromagnetic field.

A. Widom, *Quantum Electrodynamic Circuits at Ultralow Temperature*, **Journal Journal of Low Temperature Physics**, Vol. 37, Nos. 3/4, 1979; A Widom, TD Clark, *Quantum electrodynamic uncertainty relations*, **Physics Letters A**, 90, 2801982.

Nonlinear LC Circuit



For this nonlinear circuit, the commutation relation

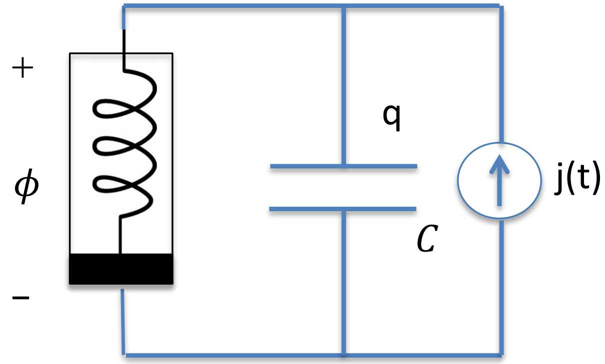
$$[\hat{\phi}, \hat{q}] = i\hbar$$

can be derived from the physical model of the Josephson junction.

P. W. Anderson, in Lectures on the Many-Body Problem, E. R. Caianiello, ed. (Academic Press, New York, 1964), Vol. 2, p. 113.

Uri Vool, Michel Devoret, *Introduction to quantum electromagnetic circuits*, **Int. J. Circ. Theor. Appl.** 2017; 45:897–934.

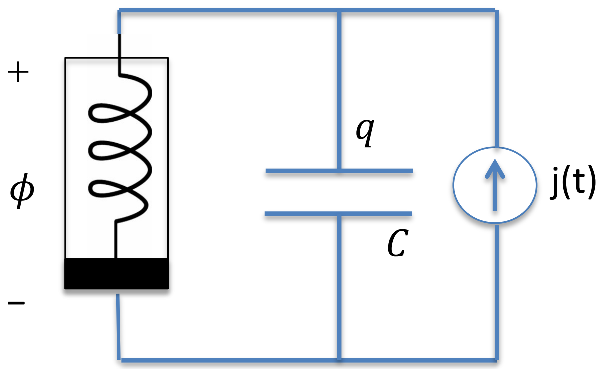
Quantization of Electrical Circuits



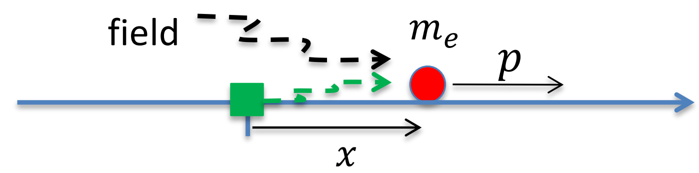
In general, the conjugate observables $\hat{\phi}$ and $\hat{q}$ do not commutate: they verify the commutation relation

$$[\hat{\phi}, \hat{q}] = i\hbar.$$

Quantum electrical circuit versus 1-D quantum electron motion



Non linear "LC circuit"



Electron under the action of an electric field

Canonical conjugated physical variables (q,ϕ)

Fundamental Observables
$q \rightarrow \hat{q}, \phi \rightarrow \hat{\phi}$

The physical variables q and ϕ are incompatible

Commutation Relation
$[\hat{\phi}, \hat{q}] = i\hbar$

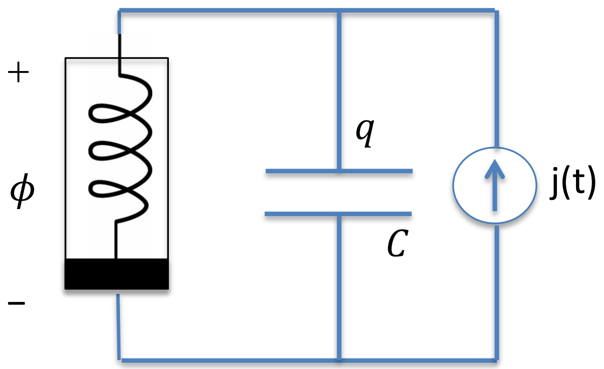
Canonical conjugated physical variables (p,x)

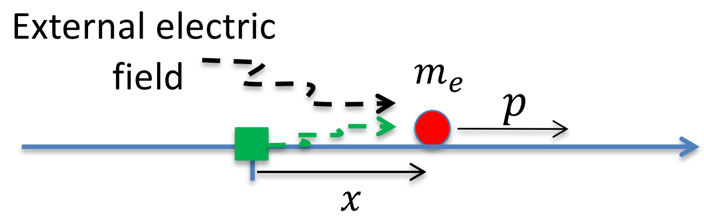
Fundamental Observables
$p \rightarrow \hat{P}, x \rightarrow \hat{X}$

The physical variables p and x are incompatible

Commutation Relation
$[\hat{X}, \hat{P}] = i\hbar$

Quantum electrical circuit versus 1-D quantum electron motion





$$\hat{\phi} = \phi, \hat{q} = -i\hbar\frac{\partial}{\partial\phi}$$

$$\hat{X} = x, \hat{P} = -i\hbar\frac{\partial}{\partial x}$$

$$\hat{H} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + W_I(\phi) - j(t)\phi$$

$$\hat{H} = -\frac{\hbar^2}{2m_e}\frac{\partial^2}{\partial x^2} + W_a(x) - F(t)x$$

$$\Psi = \Psi(\phi; t)$$

$$\psi = \psi(x; t)$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$