SCUOLA NAZIONALE DOTTORANDI DI ELETTROTECNICA
"FERDINANDO GASPARINI"

XXVII Stage

Introduction to Circuit Quantum Electrodynamics

Carlo Forestiere, Giovanni Miano

Dipartimento di Ingegneria Elettrica e delle Tecnologie dell'Informazione
Università degli Studi di Napoli Federico II

Napoli, 3 - 7 Febbraio 2025

Lecture Outline

A. Blais et al., Circuit quantum electrodynamics, Reviews of Modern Physics 93, April-June 2021.

S.E. Rasmussen et al., **Superconducting Circuit Companion—an Introduction with Worked Examples,** PRX Quantum **2**, 040204, **2021.**

A. Ciani, D. P. DiVincenzo, B. M. Terhal, **Lecture Notes on Quantum Electrical Circuits, 2024,** arXiv:2312.05329.

Superconducting Quantum Circuits

Flux representation

$\hat{\phi} = \phi, \hat{q} = -i\hbar \frac{\partial}{\partial \phi}$ fundamental observables

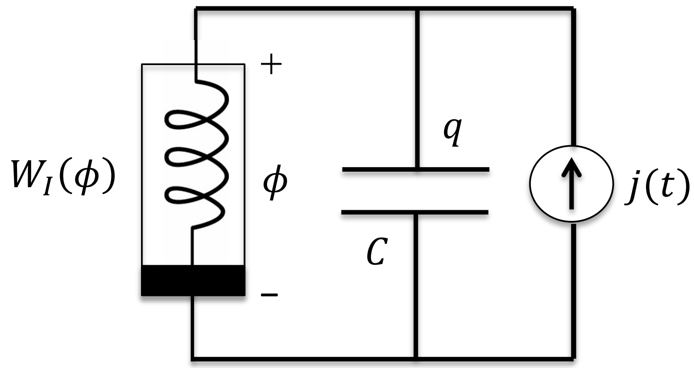
$$\hat{H} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial \phi^2} + W_I(\phi) - j(t)\phi$$

$$\Psi = \Psi(\phi; t)$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

Initial conditions: $\Psi(\phi; t = t_0) = \Psi_0\ (\phi)$

Boundary conditions: $\Psi(\phi; t)$ «regular» for $\phi \to \pm\infty$ or periodic in ϕ



First hints to quantum circuits

Quantum Electrodynamic Circuits at Ultralow Temperature

Allan Widom

Department of Physics, Northeastern University, Boston, Massachusetts

(Received March 8, 1979; revised May 30, 1979)

Within present low-temperature technology it is possible to construct macroscopic circuits which exhibit quantum behavior, i.e., subcircuit currents and voltages need to be treated as operators rather than numerical quantities. The general theory of "quantum circuits" is discussed with a view toward the experimental verification of quantum electrodynamics on a macroscopic scale.

1. INTRODUCTION

It is well known that electrodynamic processes at frequency ω require quantum mechanics if the temperature is sufficiently small, $T \lesssim \hbar\omega/k_B$. With present ultralow-temperature technology, macroscopic circuits at only moderately high frequency are "quantum circuits." The nature of quantum circuits is such that voltages and currents are operators rather than numerical quantities. Circuit oscillations are "quantized" into photons.

The purpose of this work is to present the general theory of quantum circuits with a view toward the experimental verification of quantum electrodynamics on a macroscopic scale. Clearly this requires an ultralow-temperature regime.

$k_B = 1.380649 \times 10^{-23}$ J · K^{-1}

Boltzmann constant

$\hbar$ =1.054571817…×10^{-34} J·s^{-1}

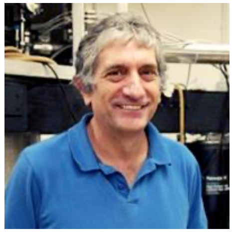
(reduced) Planck constant

$10\ mK \Leftrightarrow 208.366 \ldots MHz$

A. Widom, *Quantum Electrodynamic Circuits at Ultralow Temperature*, **Journal Journal of Low Temperature Physics**, Vol. 37, Nos. 3/4, 1979.

"Do macroscopic degrees of freedom obey quantum mechanics?"

VOLUME 55, NUMBER 15 PHYSICAL REVIEW LETTERS 7 OCTOBER 1985



J. M. Martinis

Energy-Level Quantization in the Zero-Voltage State of a Current-Biased Josephson Junction

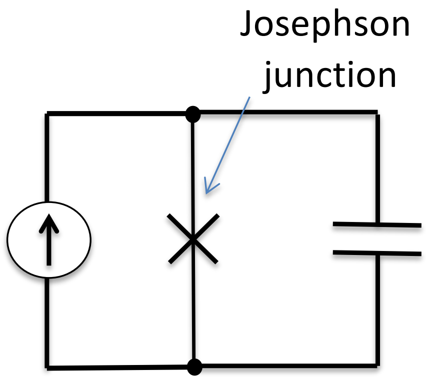
John M. Martinis, Michel H. Devoret,[(a)] and John Clarke
Department of Physics, University of California, Berkeley, California 94720, and Materials and Molecular Research Division, Lawrence Berkeley Laboratory, Berkeley, California 94720
(Received 14 June 1985)

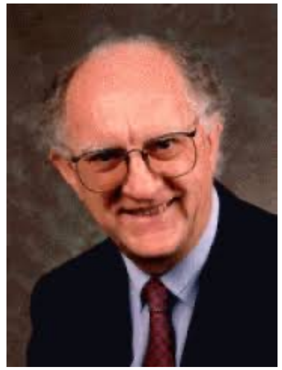
M. H. Devoret

We report the first observation of quantized energy levels for a macroscopic variable, namely the phase difference across a current-biased Josephson junction in its zero-voltage state. The position of these energy levels is in quantitative agreement with a quantum mechanical calculation based on parameters of the junction that are measured in the classical regime.

PACS numbers: 03.65.−w, 05.30.−d, 74.50.+r

Do macroscopic variables obey quantum mechanics? This question, although central to the theory of measurement,[1] has only recently been addressed experimentally. An attractive candidate for such experimental investigation is the Josephson tunnel junction, a system in which thermal fluctuations and perturbations due to the environment can be made negligible. In the case of the current-biased junction, the macroscopic variable is the phase difference, δ, between the superconducting order parameters on either side of the barrier. The junction can be represented as a particle moving in a one-dimensional tilted cosine potential.[2]





J. Clarke

[1] A. J. Legget, **Macroscopic quantum systems and the quantum theory of measurement**, Progress of Theoretical Physics Supplement, 1980.

“Macroscopic nucleus with wires”

Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction

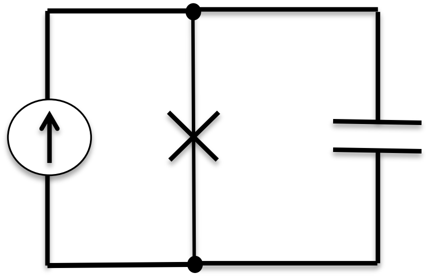
JOHN CLARKE, ANDREW N. CLELAND, MICHEL H. DEVORET, DANIEL ESTEVE, JOHN M. MARTINIS

Experiments to investigate the quantum behavior of a macroscopic degree of freedom, namely the phase difference across a Josephson tunnel junction, are described. The experiments involve measurements of the escape rate of the junction from its zero voltage state. Low temperature measurements of the escape rate for junctions that are either nearly undamped or moderately damped agree very closely with predictions for macroscopic quantum tunneling, with no adjustable parameters. Microwave spectroscopy reveals quantized energy levels in the potential well of the junction in excellent agreement with quantum-mechanical calculations. The system can be regarded as a “macroscopic nucleus with wires.”

ARE MACROSCOPIC DEGREES OF FREEDOM GOVERNED BY quantum mechanics? Our everyday experience tells us that a classical description appears to be entirely adequate. The trajectory of the center of mass of a billiard ball is predicted wonderfully well by classical mechanics. Even the Brownian motion of a tiny speck of dust in a drop of water is a purely classical phenomenon. Until recently, quantum mechanics manifested itself at the macroscopic level only through such collective phenomena as superconductivity, flux quantization, or the Josephson effect. However, these “macroscopic” effects actually arise from the coherent superposition of a large number of microscopic variables each governed by quantum mechanics. Thus, for example, the current through a Josephson tunnel junction and the phase difference across it are normally treated as classical variables. As Leggett (*1*) has emphasized, one must distinguish carefully between macroscopic quantum phenomena originating in the superposition of a large number of microscopic variables and those displayed by a single macroscopic degree of freedom. It is the latter that we discuss in this article.

Our usual observations on a billiard ball or Brownian particle reveal classical behavior because Planck's constant $\hbar$ is so tiny. However, at least in principle there is nothing to prevent us from designing an experiment in which these objects are quantum mechanical. To do so we have to satisfy two criteria: (i) the thermal energy must be small compared with the separation of the quantized energy levels, and (ii) the macroscopic degree of freedom must be sufficiently decoupled from all other degrees of freedom if the lifetime of the quantum states is to be longer than the characteristic time scale of the system (*1*). To illustrate the application of these criteria, following Leggett (*1*) we consider a simple harmonic oscillator consisting of an inductor L connected in parallel with a capacitor C. The flux Φ in the inductor and charge q on the capacitor are macroscopic conjugate variables. Observations on the oscillator are made by means of leads that unavoidably couple it to the environment. The dissipation so introduced is represented by a resistor R in parallel with L and C. The natural angular frequency of oscillation is $\omega_0 = (LC)^{-1/2}$, the impedance at the resonance frequency is $Z_0 = (L/C)^{1/2}$, and the quality factor (ratio of stored energy to energy dissipated in one oscillation) is $Q = \omega_0 CR = R/Z_0$. To observe quantum effects we thus require (i) $\hbar\omega_0 >> k_B T$, where

J. Clarke and A. N. Cleland are in the Department of Physics, University of California, and the Materials and Chemical Sciences Division, Lawrence Berkeley Laboratory, Berkeley, CA 94720. During the time these experiments were performed M. H. Devoret and J. M. Martinis were at the same address; they and D. Esteve are currently at Service de Physique, Centre d'Études Nucléaires de Saclay, 91191 Gif-sur-Yvette Cedex, France.

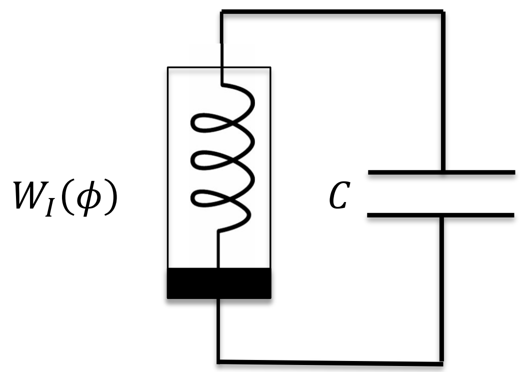


Science, vol. 239, **1988**

3.1.1 Artificial atoms: stationary states

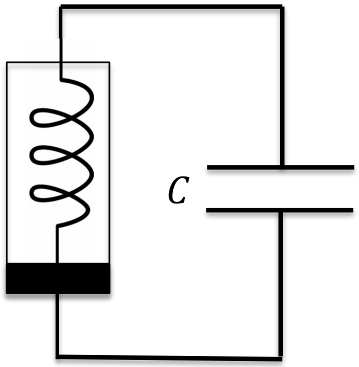
Isolated Superconducting Quantum Circuit

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2C} \frac{\partial^2}{\partial \phi^2} + W_I(\phi) \right] \Psi\,(\phi; t)$$



Energy Eigenstates

$$\hat{E} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + W_I(\phi)$$

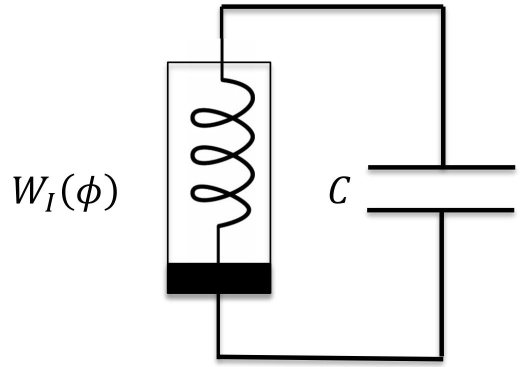


The energy eigenstates are solutions of the eigenvalue problem

$$\hat{E}\ \xi_E(\phi) = E\xi_E(\phi)$$

Stationary States

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[-\frac{\hbar^2}{2C} \frac{\partial^2}{\partial \phi^2} + W_I(\phi) \right] \Psi\,(\phi; t)$$



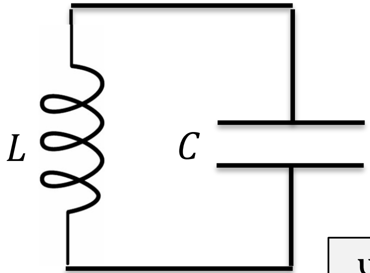
The wave function

$$\Psi(\phi; t) = e^{-iEt/\hbar} \xi_E(\phi)$$

is solution of the time independent $\mathrm{Schr\ddot{o}dinger}$ equation.

It represents a **stationary state** of the quantum circuit: the probaibility density $|\Psi(\phi; t)|^2 = |\xi_E(\phi)|^2$ is constant in time.

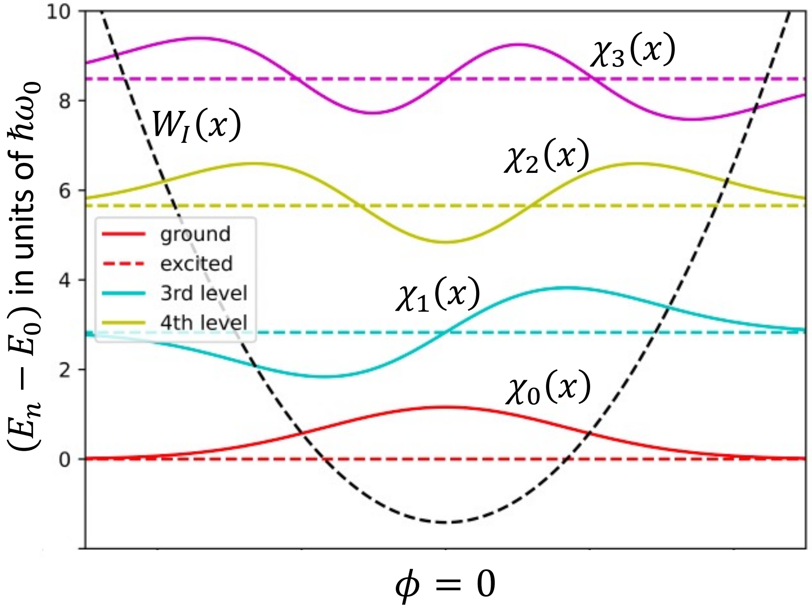
Stationary States



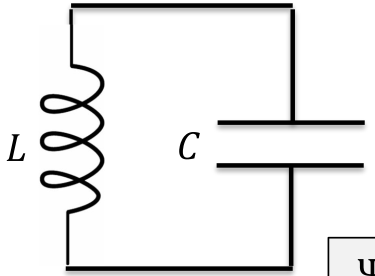
$$\hat{E} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2$$

$$\Psi_n(\phi; t) = e^{-iE_n t/\hbar}\chi_n(\phi)$$

All eigenstates are bound

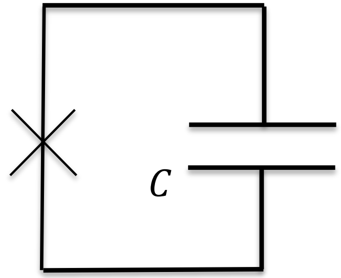


Stationary States



$$\hat{E} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2$$

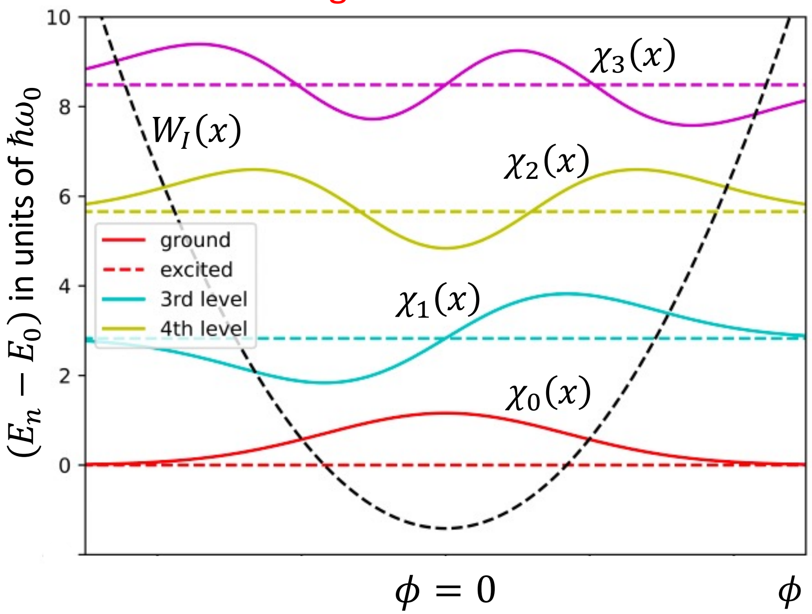
$$\Psi_n(\phi;t) = e^{-iE_nt/\hbar}\chi_n(\phi)$$



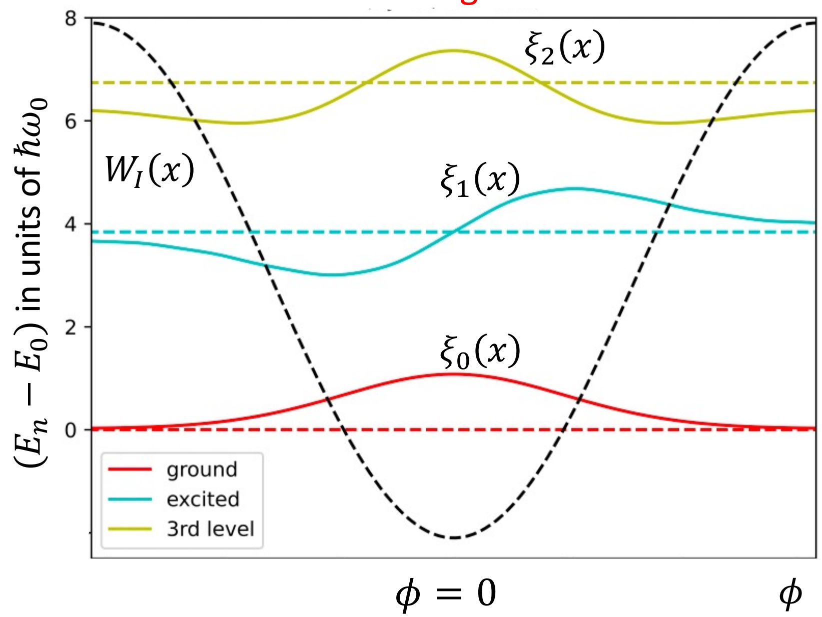
$$\hat{E} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + E_J[1 - cos(2\pi\phi/\Phi_0)]$$

$$\Psi_n(\phi;t) = e^{-iE_nt/\hbar}\xi_n(\phi)$$

All eigenstates are bound

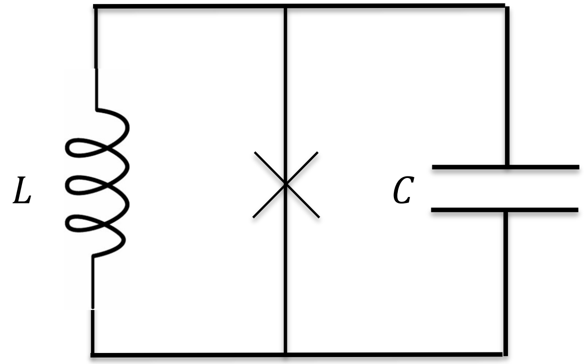


bound eigenstates

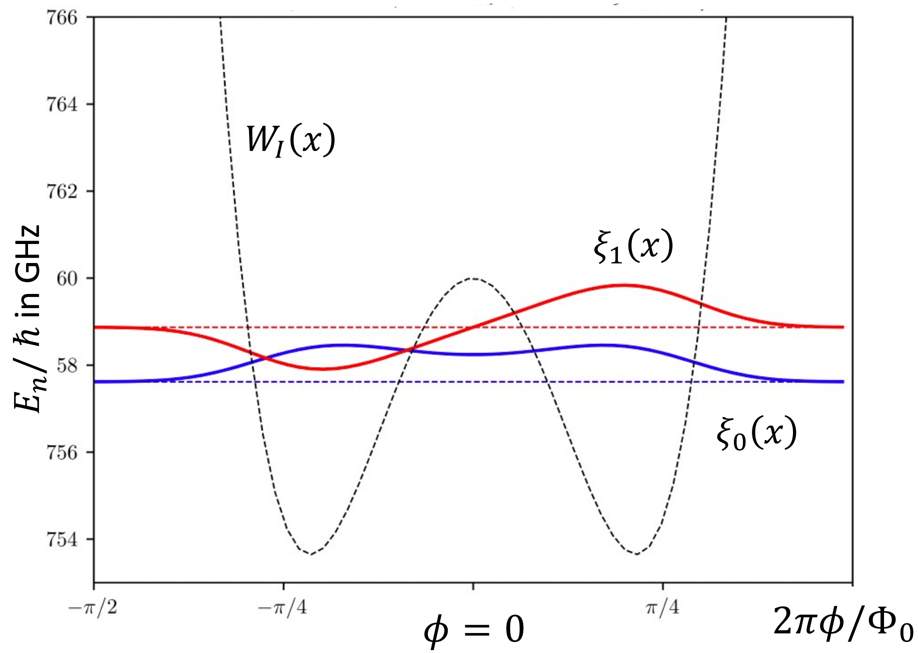


Stationary States

$$\hat{E} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + E_J\,[1 - cos(2\pi\phi/\Phi_0)] + \frac{1}{2L}\phi^2$$



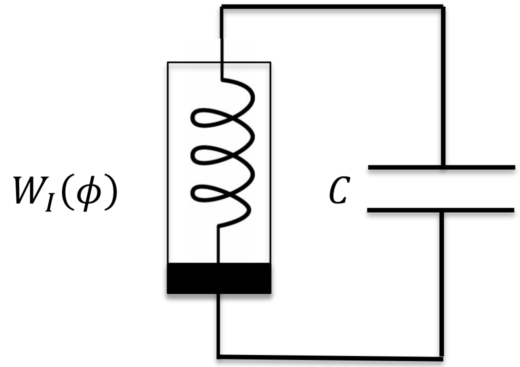
All eigenstates are bound



$$\Psi_n(\phi;t) = e^{-iE_n t/\hbar}\xi_n(\phi)$$

Superposition of Stationary States

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[-\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + W_I(\phi)\right]\Psi(\phi;t)$$



$$\Psi\,(\phi;t) = \sum_n c_n\,\xi_n(\phi)e^{-i\omega_n t} \text{ where } \omega_n = E_n/\,\hbar \text{ and } n = 0,1,2,\ldots$$

$$j(t) = 0$$

The probability that the measurement of the energy yields the value E_n when the circuit is in the state $\Psi(\phi;t)$,

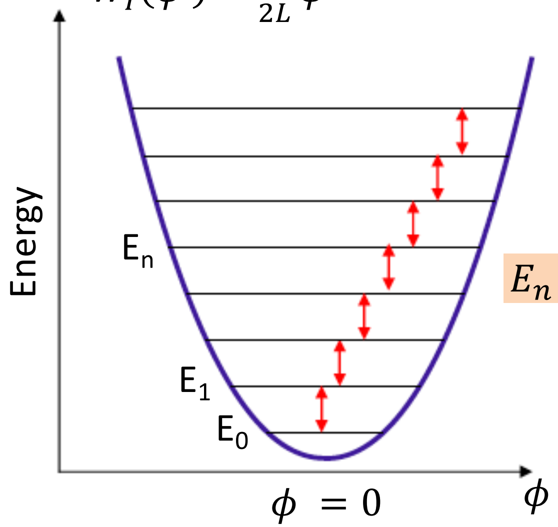
$$P_E(E_n|\Psi) = |\langle\xi_n|\Psi\rangle|^2 = |c_n|^2.$$

3.1.2 Artificial atoms: Transmon

Resonator versus Artificial Atom

Linear Resonator

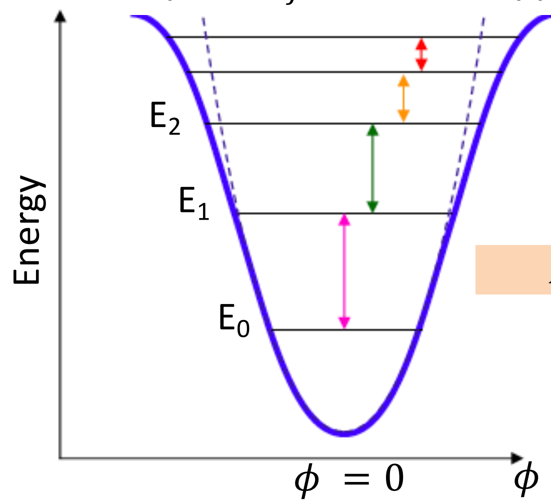
$W_I(\phi) = \frac{1}{2L}\phi^2$



$E_n - E_{n-1} = \hbar\omega_e$

Artificial Atom

$W_I(\phi) = E_J[1 - cos(2\pi\phi/\Phi_0)]$



$E_1 - E_0 > E_2 - E_1$

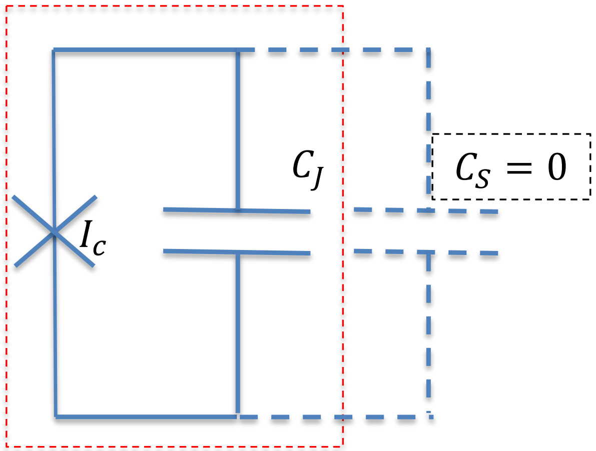
The energy levels of the harmonic oscillator are uniformly spaced.

Anharmonicity gives arise to non uniformity in the distribution of energy levels.

The nonlinearity of Josephson junction allows the realization of artificial atom.

Artificial Atom: Two different scenarios

Josephson junction



Cooper pair box: $\frac{E_J}{E_C} \ll 1$

$$E_C = \frac{e^2}{2C_J}$$

Cooper pair box: the “artificial atom” is highly sensitive to charge noise, which has proven more challenging to mitigate than flux noise, making it very hard to achieve high coherence.

Jens Koch, Terri M. Yu, Jay Gambetta, A. A. Houck, D. I. Schuster, J. Majer, Alexandre Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, *Charge-insensitive qubit design derived from the Cooper pair box*, Physical Review A, 2007.

Artificial Atom: Two different scenarios

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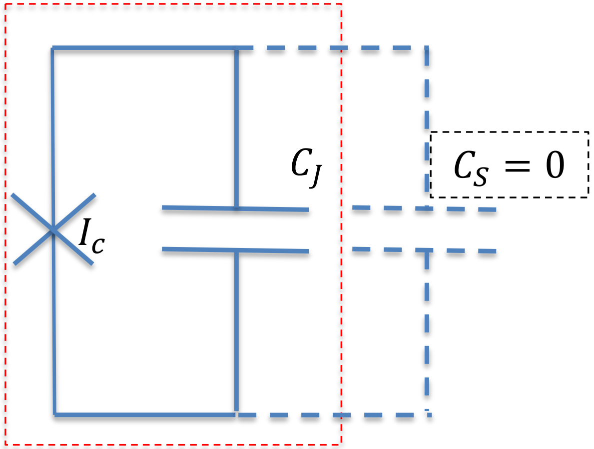
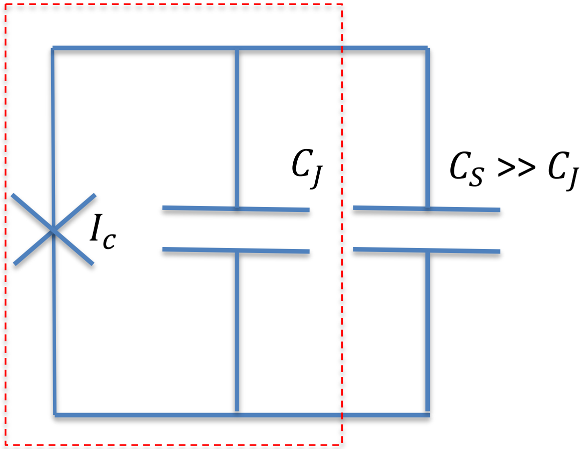


Cooper pair box: the "artificial atom" is highly sensitive to charge noise, which has proven more challenging to mitigate than flux noise, making it very hard to achieve high coherence.

Josephson junction

Transmon: $\frac{E_J}{E_C} \gg 1$

$$E_C = \frac{e^2}{2C_t}, C_t = (C_J + C_S)$$



Transmon: the "artificial atom" is insensitive to charge noise.

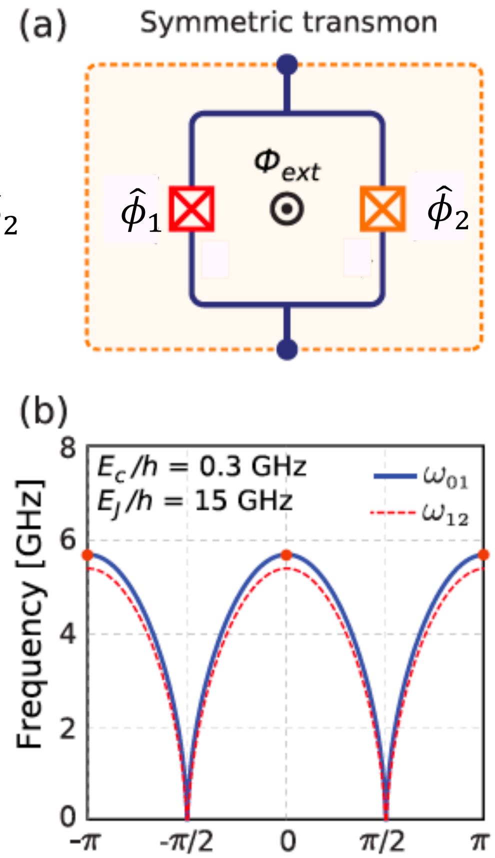
$$E_{01} \equiv E_1 - E_0 \cong \hbar\omega_p - E_C,$$

$$\omega_p = \frac{1}{\hbar}\sqrt{8E_C|E_J\ |}$$

Jens Koch, Terri M. Yu, Jay Gambetta, A. A. Houck, D. I. Schuster, J. Majer, Alexandre Blais, M. H. Devoret, S. M. Girvin, and R. J. Schoelkopf, *Charge-insensitive qubit design derived from the Cooper pair box*, Physical Review A, 2007.

Split transmon

$\hat{\phi}_1 \neq \hat{\phi}_2$



$$\hat{E}_a(\hat{q},\hat{\phi}_1) = \frac{\hat{q}^2}{2C} + E_J'\left[cos\left(\frac{\pi\Phi_e}{\Phi_0}\right) - cos\left(2\pi\frac{\phi_1}{\Phi_0} + \frac{\pi\Phi_{ext}}{\Phi_0}\right)\right]$$

$$E_J' = E_J\cos(\pi\Phi_{ext}/\Phi_0)$$

$$E_{01} \cong \hbar\omega_p - E_C, \omega_p = \frac{1}{\hbar}\sqrt{8E_C|E_J'|}$$

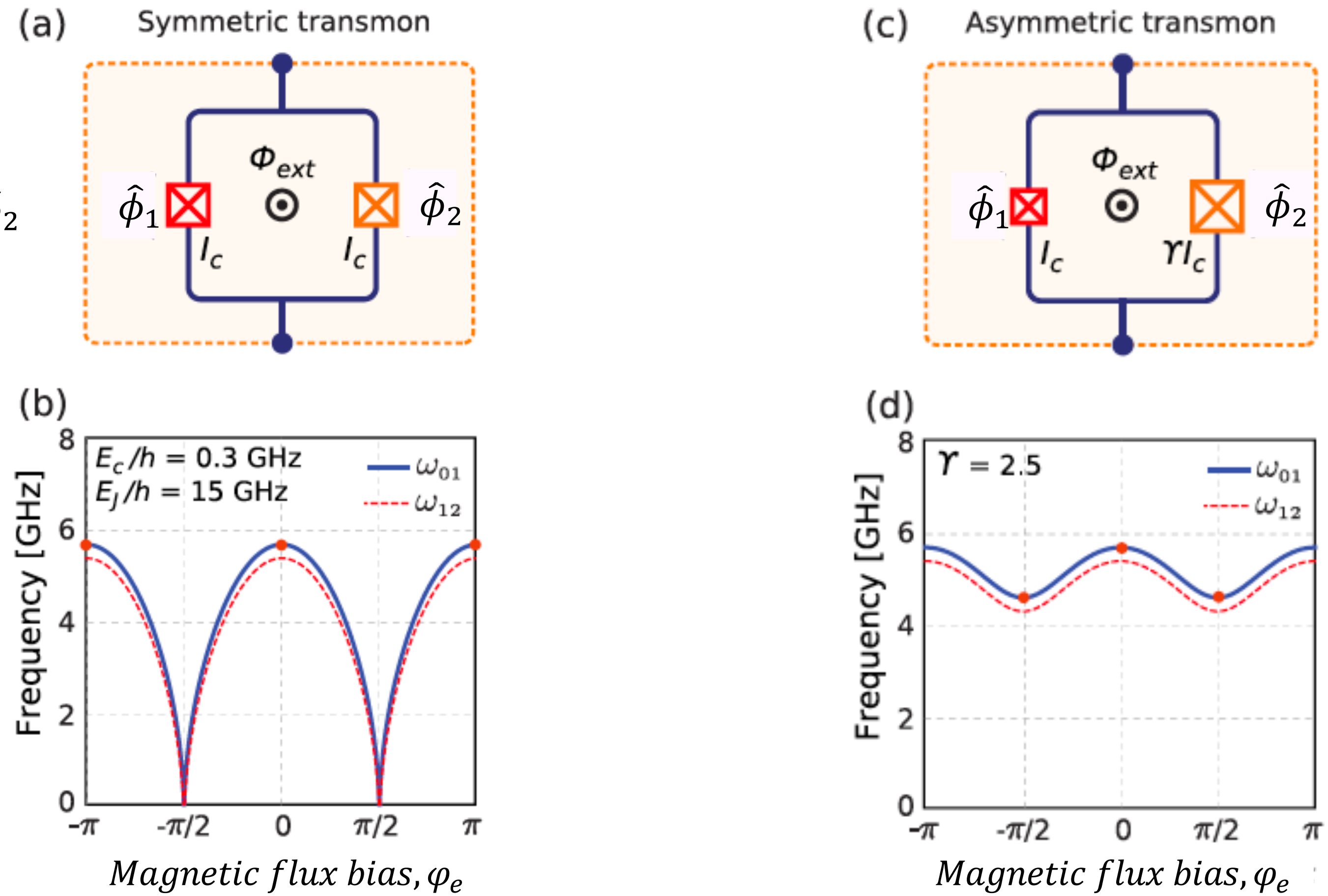
$$\omega_{01} = \frac{E_{01}}{\hbar}, \omega_{12} = \frac{E_{12}}{\hbar}$$

Magnetic flux bias, $\varphi_e = \pi\Phi_e/\Phi_0$

P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver, *A quantum engineer's guide to superconducting qubits*, Applied Physics Review, 2019.

Split transmon

$\hat{\phi}_1 \neq \hat{\phi}_2$



P. Krantz, M. Kjaergaard, F. Yan, T. P. Orlando, S. Gustavsson, and W. D. Oliver, *A quantum engineer's guide to superconducting qubits*, Applied Physics Review, 2019.

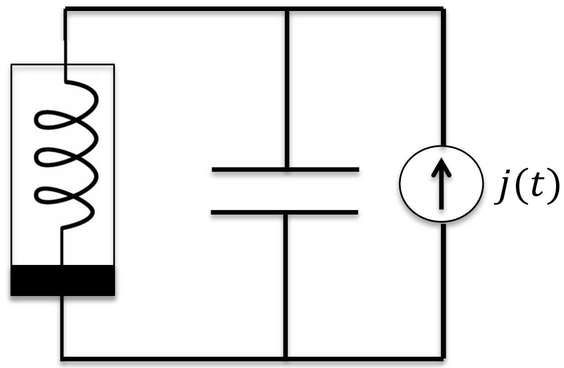
3.1.3 Artificial atoms: Transition Probabilities

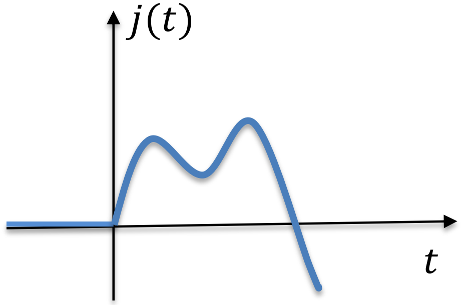
Transition probability

The circuit is assumed to be in the stationary state $\xi_i(\phi)e^{i\omega_i t}$ for $t < 0$, e.g., $j(t) = 0$ for $t < 0$.

At $t = 0$ the current generator switches on, and for $t > 0$ the quantum state of circuit is no longer the stationary state $\xi_i(\phi)e^{i\omega_i t}$.

We introduce the probability $P_{if}(t)$ of finding the circuit in another stationary state $\xi_f(\phi)e^{i\omega_f t}$ at time t, in other words, we want to study the transitions induced by the current generator between the stationary states $\xi_i(\phi)e^{i\omega_i t}$ (initial state) and the stationary state $\xi_f(\phi)e^{i\omega_f t}$ (final state).



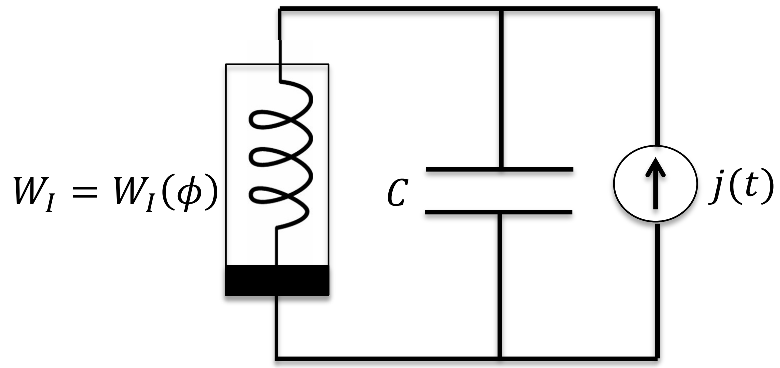


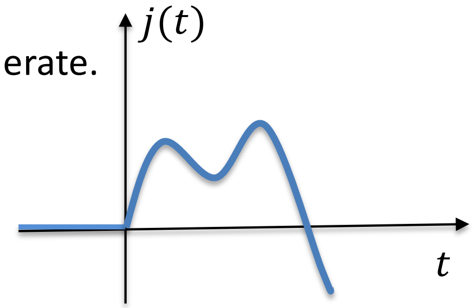
$$\begin{cases} \Psi_0(\phi) = \xi_i(\phi) \\ \wp_{i\to f}(t) = \left|\langle \xi_f | \Psi \rangle\right|^2 = \left| \int_{-max}^{+\phi_{max}} \xi_f(\phi)\, \Psi(\phi;t)\, d\phi \right|^2 \end{cases}$$

C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. II, 2nd ed., Wiley-VCH, 2019.

Artificial Atom Driven by a Time Varying Current Source

$$\begin{cases} i\hbar\dfrac{\partial\Psi}{\partial t} = \overbrace{\left[-\dfrac{\hbar^2}{2C}\dfrac{\partial^2}{\partial\phi^2} + W_I(\phi)\right]}^{\hat{E}}\Psi \underbrace{- j(t)\phi\Psi}_{\text{Source term}} \\ \Psi(\phi; t = t_0) = \Psi_0\,(\phi) \end{cases}$$





We assume that the spectrum of the energy of the circuit $\hat{E}$ is discrete and not degenerate.

The energy eigenstates are solutions of the eigenvalue problem

$$\hat{E}\,\xi_n(\phi) = E_n\xi_n(\phi),\, n = 0,1,2,\ldots$$

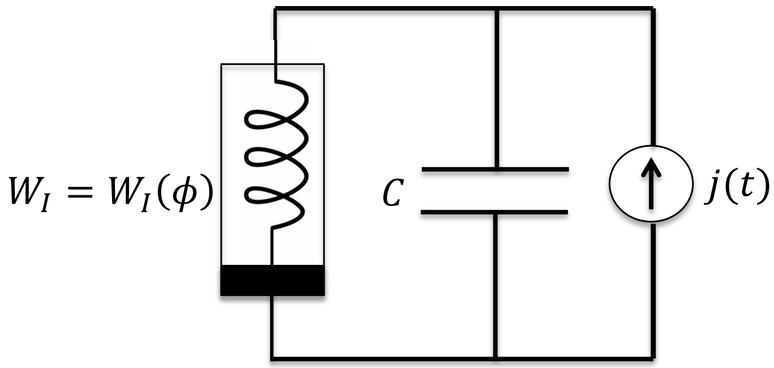
where

$$\langle\xi_m|\xi_n\rangle = \delta_{mn}.$$

We denote with $(-\phi_{max}, +\phi_{max})$ the support of the eigenfunctions $\xi_n(\phi)$.

Artificial Atom Driven by a Time Varying Current Source

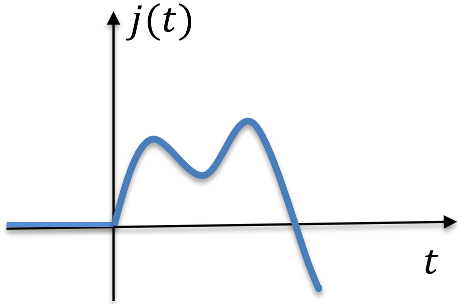
$$\begin{cases} i\hbar \dfrac{\partial \Psi}{\partial t} = \overbrace{\left[-\dfrac{\hbar^2}{2C}\dfrac{\partial^2}{\partial \phi^2} + W_I(\phi) \right]}^{\hat{E}} \Psi \underbrace{- j(t)\phi\Psi}_{\text{Source term}} \\ \Psi(\phi; t = t_0) = \Psi_0(\phi) \end{cases}$$



The solution of the time-dependent Schrödinger equation can represented as a superposition of the stationary states $\{\xi_n(\phi)e^{-i\omega_n t}\}$,

$$\Psi\,(\phi; t) = \sum_n c_n(t)\xi_n(\phi)e^{-i\omega_n t} \text{ where } \omega_n = E_n/\,\hbar \text{ and } n = 0,1,2, \ldots .$$

The expansion coefficients are functions of time because the Hamiltonian depends explicitly on time through the interaction term $-j(t)\phi$.



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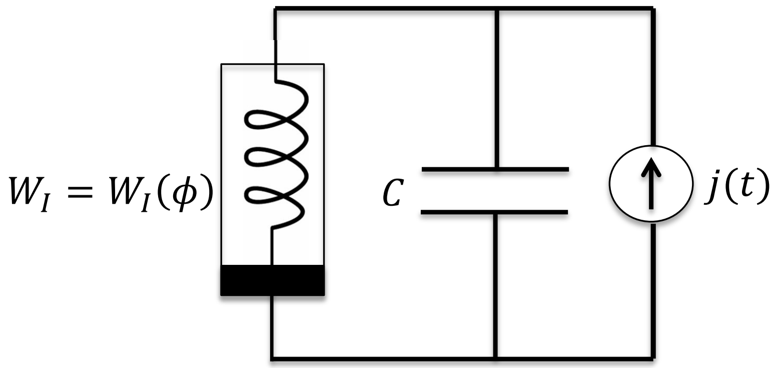
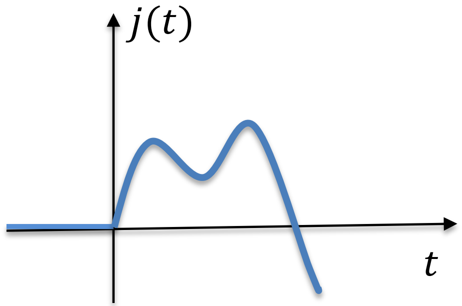


The solution of the time-dependent Schrödinger equation can represented as a superposition of the stationary states $\{\xi_n(\phi)e^{-i\omega_n t}\}$,

$$\Psi\,(\phi; t) = \sum_n c_n(t)\xi_n(\phi)e^{-i\omega_n t} \text{ where } \omega_n = E_n/\,\hbar \text{ and } n = 0,1,2, \dots .$$

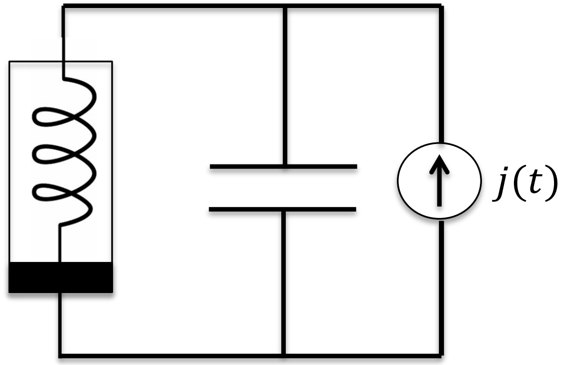


Substituting this expression into the Schrödinger equation, given that $\xi_n(\phi)$ is the eigenfunction of the energy observable with eigenvalue E_n, and using the orthonormality property of the eigenfunctions, we obtain for $c_m(t)$ the ordinary differential equation

$$\dot{c}_m = \frac{1}{i\hbar}\sum_n W_{mn}(t)\, c_n(t)e^{i\omega_{mn}t} \text{ for } m = 0,1,2, \dots$$

where $\omega_{mn} = (E_m - E_n)/\hbar$, $W_{mn}(t) = j(t)\Phi_{mn}$ and $\Phi_{mn} = \langle\xi_m|\phi\xi_n\rangle$. The equations are solved with initial conditions $c_m(0) = \langle\xi_m|\Psi_0\rangle$.

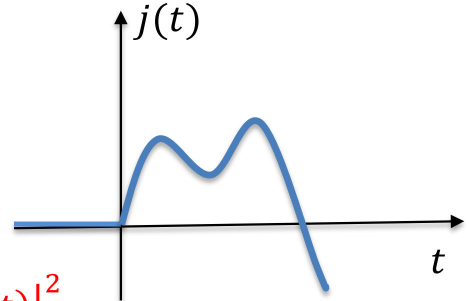
Transition probability



The circuit is assumed to be in the stationary state $\xi_i(\phi)e^{i\omega_i t}$ for $t < 0$.

At $t = 0$ the current generator is applied, and for $t > 0$ the quantum state of circuit is no longer the stationary state $\xi_i(\phi)e^{i\omega_i t}$.

We introduce the probability $P_{if}(t)$ of finding the circuit in another stationary state $\xi_f(\phi)e^{i\omega_f t}$ at time t, in other words, we want to study the transitions induced by the current generator between the stationary states $\xi_i(\phi)e^{i\omega_i t}$ (initial state) and the stationary state $\xi_f(\phi)e^{i\omega_f t}$ (final state),



$$\begin{cases} \Psi_0(\phi) = \xi_i(\phi) \\ \wp_{i\to f}(t) = \left|\langle \xi_f | \Psi \rangle\right|^2 = \left| \int_{-max}^{+\phi_{max}} \xi_f(\phi)\, \Psi(\phi; t) d\phi \right|^2 = \left|c_f(t)\right|^2 \end{cases}$$

C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. II, 2nd ed., Wiley-VCH, 2019.

Transition probability

When

$$j(t) = J_m cos(\omega t)$$

then

$$\wp_{i\to f}(t) \cong \left(\frac{J_m \Phi_{if}}{\hbar}\right)^2 \left\{\frac{sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}\right\}^2 \text{ for } \omega \approx \omega_{fi} \text{ and } \frac{1}{\omega_{fi}} \ll t \ll \frac{\hbar}{J_m \Phi_{if}}$$

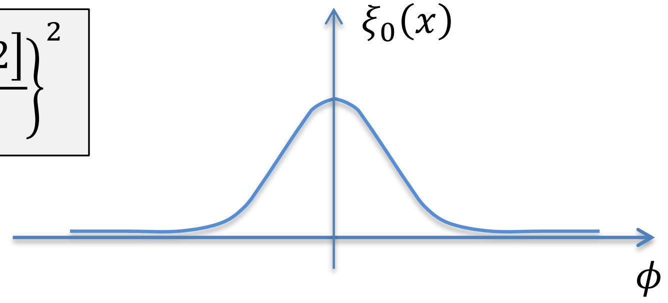
where

$$\omega_{fi} = \frac{E_f - E_i}{\hbar}, \Phi_{if} = \langle \xi_i | \phi \xi_f \rangle = \int_{-\phi_{max}}^{+\phi_{max}} d\phi \ \xi_i(\phi) \phi \xi_f(\phi).$$

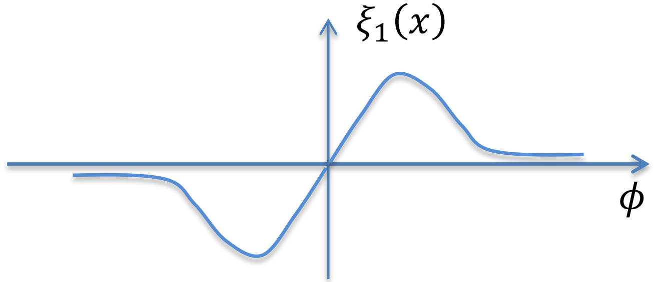
C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, II, 2nd ed., Wiley-VCH, 2019.

Selection Rules

$$\wp_{i\to f}(t) \cong \left(\frac{J_m \Phi_{if}}{\hbar}\right)^2 \left\{\frac{sin[(\omega_{fi}-\omega)t/2]}{(\omega_{fi}-\omega)/2}\right\}^2$$

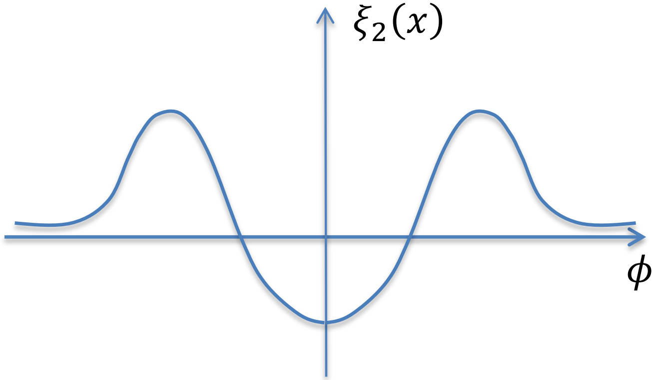


$$\Phi_{if} = \int_{-\phi_{max}}^{+\phi_{max}} d\phi\ \xi_m(\phi)\phi\xi_n(\phi)$$



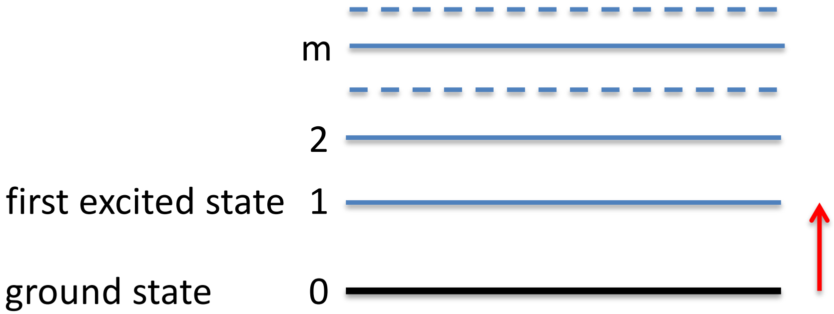
$$\Phi_{01} = \int_{-\phi_{max}}^{+\phi_{max}} \xi_0(\phi)\,\phi\xi_1(\phi)\,d\phi \neq 0$$

Transitions from the ground state to excited states with m even are forbidden.

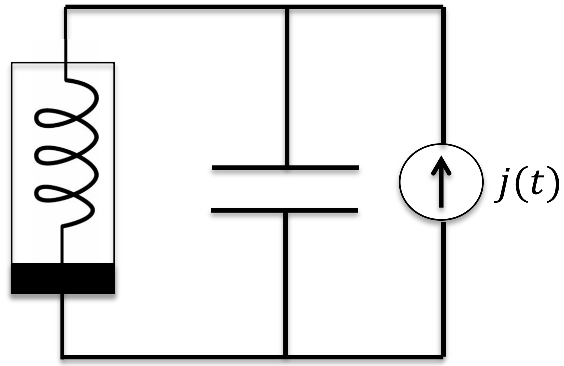


$$\Phi_{02} = \int_{-\phi_{max}}^{+\phi_{max}} \xi_0(\phi)\,\phi\xi_2(\phi)\,d\phi = 0$$

Resonant Interaction



$$\Psi_0(\phi) = \xi_0(\phi)$$



For $\omega = \omega_{10}$ we obtain a resonant interaction,

$$\wp_{0\to 1}(t) \cong \left(\frac{J_m \Phi_{10}}{\hbar}\right)^2 t^2 \text{ for } \frac{1}{\omega} \ll t \ll \frac{\hbar}{J_m \Phi_{10}}$$

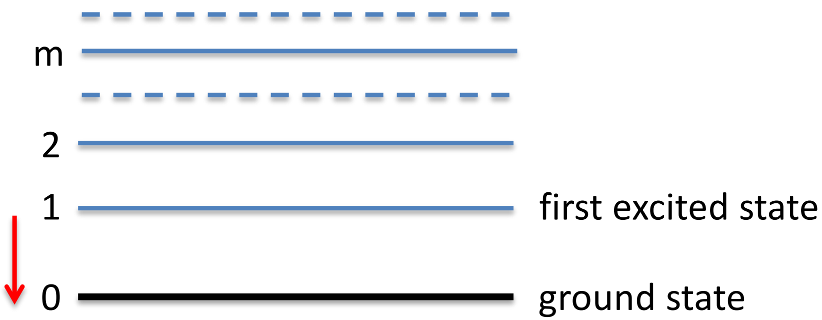
and

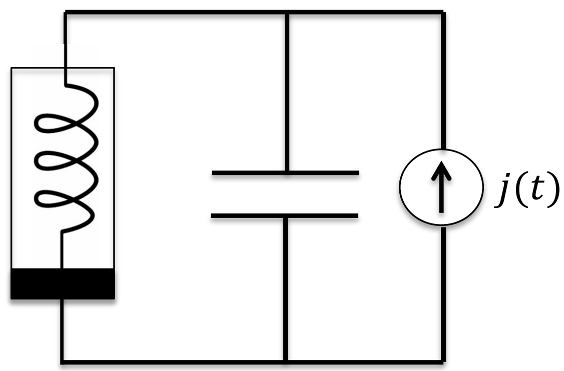
$$\wp_{0\to m}(t) \cong \left(\frac{J_m \Phi_{0m}}{\hbar}\right)^2 \left\{\frac{sin\left[\frac{(\omega_{m0}-\omega)t}{2}\right]}{\frac{(\omega_{m0}-\omega)}{2}}\right\}^2 \cong 0 \text{ for } m \geq 2.$$

Transition from the ground state to the first excited state occurs through the absorption of an energy quantum equal to $\hbar\omega_{10}$.

Resonant Condition

$\Psi_0\ (\phi) = \xi_1(\phi)$





For $\omega = \omega_{10}$ we obtain:

$$\wp_{1\to 0}(t) \cong \left(\frac{J_m \Phi_{10}}{\hbar}\right)^2 t^2 \text{ for } \frac{1}{\omega} \ll t \ll \frac{\hbar}{J_m \Phi_{10}}$$

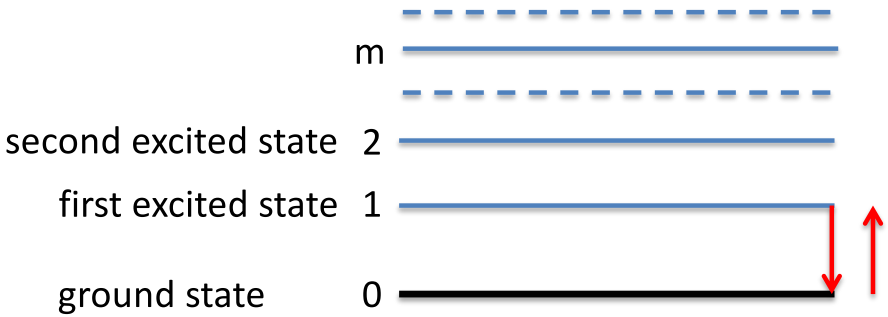
and

$$\wp_{1\to m}(t) \cong \left(\frac{J_m \Phi_{m0}}{\hbar}\right)^2 \left\{\frac{sin\left[\frac{(\omega_{m0}-\omega)t}{2}\right]}{\frac{(\omega_{m0}-\omega)}{2}}\right\}^2 \cong 0 \text{ for } m \geq 2$$

Transition from the first excited state to the ground state occurs through the emission of an energy quantum equal to $\hbar\omega_{10}$.

Resonant Condition

Transmon



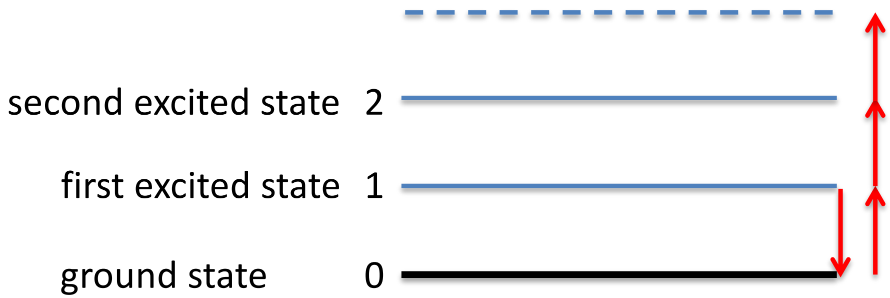
$$\omega \neq \frac{E_2 - E_1}{\hbar}$$

$$\omega = \frac{E_1 - E_0}{\hbar}$$

$$\Psi_0(\phi) = \xi_0(\phi)$$

Non uniformly spaced energy levels

Linear LC circuit



$$\omega = \frac{E_2 - E_1}{\hbar}$$

$$\omega = \frac{E_1 - E_0}{\hbar}$$

$$\Psi_0(\phi) = \xi_0(\phi)$$

Uniformly spaced energy levels

3.1.4 Artificial atoms: Two level approximation, Qubit, Rabi oscillations and Control

Two-level approximation

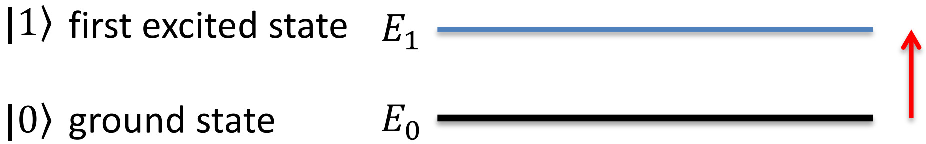
$$j(t) = J_m cos(\omega t)$$

$$\omega_{10} = \frac{E_1 - E_0}{\hbar}$$

If $\omega \cong \omega_{10}$ the transition probability $\wp_{0\to m}(t) = |c_m(t)|^2$ for $m > 1$ is negligible with respect to $\wp_{0\to 1}(t) \ = |c_1(t)|^2$. Analogously, the transition probability $\wp_{1\to m}(t) = |c_m(t)|^2$ for $m > 1$ is negligible with respect to $\wp_{1\to 0}(t) = |c_0(t)|^2$.

Therefore, for describing the time evolution of the atom when $\omega \cong \omega_{10}$ it is sound to assume $c_m(t) \cong 0$ for any $m \neq 0, 1$: the artificial atom behaves as its state space has two dimensions, namely, the artificial atom behaves as two-level system.

Qubit



$$\omega_{10} = \frac{E_1 - E_0}{\hbar}$$

The artificial atom that behaves as a two-level system implements a qubit: the computational space consists of the two eigenstates of the atom energy.

Time Evolution of the Qubit State

$$\Psi(\phi;t) \cong c_0(t)\xi_0(\phi)e^{-i\omega_0 t} + c_1(t)\xi_1(\phi)\, e^{-i\omega_1 t}$$

Disregarding all the terms with $m \geq 2$ in the expression of the wave function, we obtain

$$\begin{cases} \dot{c}_0 = \dfrac{1}{i\hbar} W_{00}(t)c_0(t)e^{i\omega_{00}t} + \dfrac{1}{i\hbar} W_{01}(t)c_1(t)e^{i\omega_{01}t} \\ \dot{c}_1 = \dfrac{1}{i\hbar} W_{10}(t)c_0(t)e^{i\omega_{10}t} + \dfrac{1}{i\hbar} W_{11}(t)c_1(t)e^{i\omega_{11}t} \end{cases}$$

with $c_0(0) = \langle \xi_0 | \Psi_0 \rangle, c_1(0) = \langle \xi_1 | \Psi_0 \rangle$.

$$\begin{cases} \omega_{mn} = (E_m - E_n)/\hbar\ , \\ W_{mn}(t) = J_m \Phi_{mn}\, cos(\omega t) \\ \Phi_{mn} = \langle \xi_m | \phi \chi_n \rangle = \int_{-\phi_{max}}^{+max} d\phi\ \xi_m(\phi)\phi\xi_n(\phi) \end{cases}$$

C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. I, 2nd ed., Wiley-VCH, 2019.

Time Evolution of the Qubit State

$$\Psi(\phi;t) \cong c_0(t)\xi_0(\phi)e^{-i\omega_0 t} + c_1(t)\xi_1(\phi)\, e^{-i\omega_1 t}$$

Disregarding all the terms with $m \geq 2$ in the expression of the wave function, we obtain

$$\begin{cases} \dot{c}_0 = \dfrac{1}{i\hbar}W_{00}(t)c_0(t)e^{i\omega_{00}t} + \dfrac{1}{i\hbar}W_{01}(t)c_1(t)e^{i\omega_{01}t} \\ \dot{c}_1 = \dfrac{1}{i\hbar}W_{10}(t)c_0(t)e^{i\omega_{10}t} + \dfrac{1}{i\hbar}W_{11}(t)c_1(t)e^{i\omega_{11}t} \end{cases}$$

with $c_0(0) = \langle \xi_0|\Psi_0\rangle, c_1(0) = \langle \xi_1|\Psi_0\rangle$.

$$\begin{cases} \omega_{mn} = (E_m - E_n)/\hbar\,, \\ W_{mn}(t) = J_m\Phi_{mn}\, cos(\omega t) \\ \Phi_{mn} = \langle \xi_m|\phi\chi_n\rangle = \int_{-\phi_{max}}^{+\phi_{max}} d\phi\; \xi_m(\phi)\phi\xi_n(\phi) \end{cases}$$

$$\omega_{00} = \omega_{11} = 0, \omega_{10} = \frac{(E_1 - E_0)}{\hbar}, \omega_{01} = -\omega_{10}$$

$$\Phi_{00} = \Phi_{11} = 0, \Phi_{10} = \Phi_{01} = \int_{-\phi_{max}}^{+max} d\phi\;\; \xi_0(\phi)\phi\xi_1(\phi)$$

Time Evolution of the Qubit State

$$\Psi(\phi;t) \cong c_0(t)\xi_0(\phi)e^{-i\omega_0 t} + c_1(t)\xi_1(\phi)\, e^{-i\omega_1 t}$$

$$\begin{cases} \dot{c}_0 = \frac{J_m \Phi_{10}}{i\hbar} c_1(t)e^{-i\omega_{10}t}\, cos(\omega t) \\ \dot{c}_1 = \frac{J_m \Phi_{10}}{i\hbar} c_0(t)e^{+i\omega_{10}t}\, cos(\omega t) \end{cases}$$

with $c_0(0) = \langle \xi_0 | \Psi_0 \rangle, c_1(0) = \langle \xi_1 | \Psi_0 \rangle$

$$\omega_{10} = \frac{(E_1 - E_0)}{\hbar}, \Phi_{10} = \int_{-\phi_{max}}^{+\phi_{max}} d\phi\;\; \xi_0(\phi)\phi\xi_1(\phi)$$

$$\begin{cases} \dot{c}_0 = -i\Omega_R cos(\omega t)c_1(t)e^{-i\omega_{10}t}, \\ \dot{c}_1 = -i\Omega_R cos(\omega t)c_0(t)e^{+i\omega_{10}t}, \end{cases}$$

with $c_0(0) = \langle \xi_0 | \Psi_0 \rangle, c_1(0) = \langle \xi_1 | \Psi_0 \rangle$

$\Omega_r = \frac{J_m \Phi_{10}}{\hbar}$ is the **Rabi frequency.**

For a transom $\Phi_{10} \cong \sqrt{\frac{\hbar Z_t}{2}}$ where $Z_t = \sqrt{\frac{L_J}{C_t}}$

Resonant wave approximation

$$\Psi(\phi;t) \cong c_0(t)\xi_0(\phi)e^{-i\omega_0 t} + c_1(t)\xi_1(\phi)\, e^{-i\omega_1 t}$$

$$\begin{cases} c_0(t) = e^{i(\omega-\omega_{10})t/2}\, b_0(t), \\ c_1(t) = e^{-i(\omega-\omega_{10})t/2}\, b_1(t), \end{cases}$$

$$\begin{cases} b_0(t) \cong cos\left(\frac{\Omega_R}{2}t\right) - 2i\,\frac{(\omega-\omega_{10})}{\Omega_R} sin\left(\frac{\Omega_R}{2}t\right), \\ b_1(t) \cong -i\,\frac{\Omega_r}{\Omega_R} sin\left(\frac{\Omega_R}{2}t\right), \end{cases}$$

if $\Omega_r \ll \omega_{10}$ and $\omega \approx \omega_{10}$

where $\Omega_R(\omega) = \sqrt{(\omega-\omega_{10})^2 + \Omega_r^2}$ is the Rabi flopping frequency.

In the resonant condition $\omega = \omega_{10}$ it is $\Omega_R = \Omega_r$.

Probabilities

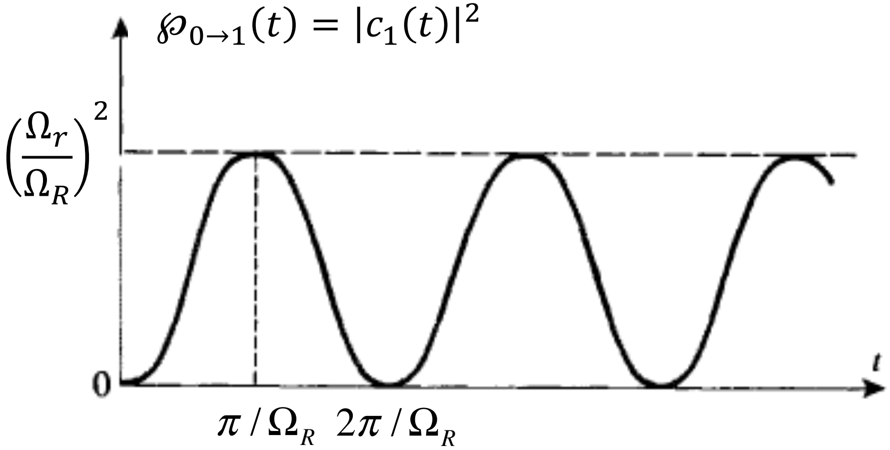
$$\omega_{10} = \frac{E_1 - E_0}{\hbar}$$

The qubit is initially in the ground state, $c_0(0) = 1, c_1(0) = 0.$

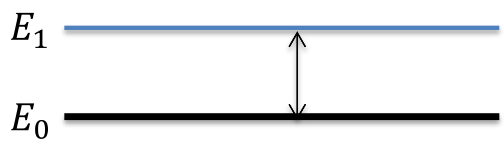
$$\begin{cases} b_0(t) \cong cos\left(\frac{\Omega_R}{2}t\right) - 2i\frac{(\omega - \omega_{10})}{\Omega_R}sin\left(\frac{\Omega_R}{2}t\right), \\ b_1(t) \cong -i\frac{\Omega_r}{\Omega_R}sin\left(\frac{\Omega_R}{2}t\right), \end{cases}$$

$$\begin{cases} |c_0(t)|^2 = |b_0(t)|^2 = 1 - \frac{\Omega_r^2}{\Omega_R^2}sin^2\left(\frac{\Omega_R}{2}t\right) \\ |c_1(t)|^2 = |b_0(t)|^2 = \frac{\Omega_r^2}{\Omega_R^2}sin^2\left(\frac{\Omega_R}{2}t\right) \end{cases} \Rightarrow |c_0(t)|^2 + |c_1(t)|^2 = 1$$

Rabi Oscillations



$$\begin{cases} |c_0(t)|^2 = 1 - \left(\frac{\Omega_r}{\Omega_R}\right)^2 sin^2\left(\frac{\Omega_R}{2}t\right) \\ |c_1(t)|^2 = \left(\frac{\Omega_r}{\Omega_R}\right)^2 sin^2\left(\frac{\Omega_R}{2}t\right) \end{cases}$$



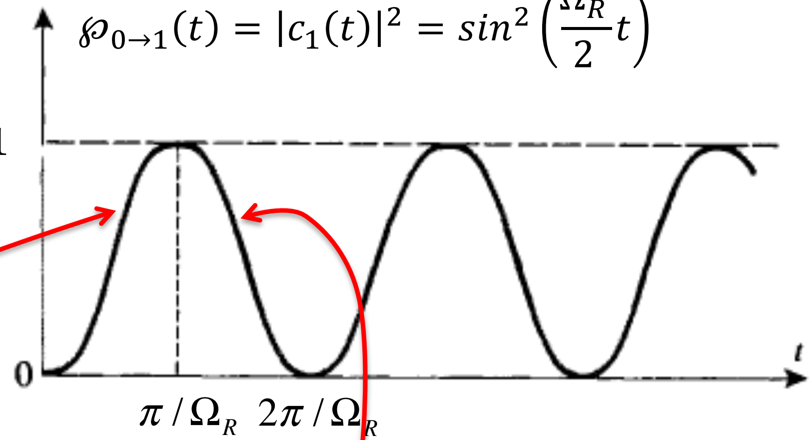
$$\omega_{10} = \frac{E_1 - E_0}{\hbar}$$

$$\Omega_R = \sqrt{(\omega - \omega_{10})^2 + \Omega_r^2}$$

C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. I, 2nd ed., Wiley-VCH, 2019.

Rabi Oscillations

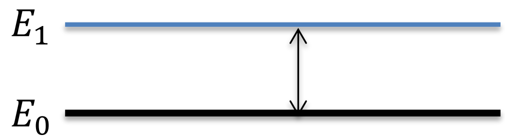
$$\wp_{0\to1}(t) = |c_1(t)|^2 = sin^2\left(\frac{\Omega_R}{2}t\right)$$



Resonant interaction $\lim_{\omega\to\omega_{10}}\left(\frac{\Omega_r}{\Omega_R}\right)^2 = 1$

the circuit goes in to the *excited state* at $t_1 \cong \pi / \Omega_R$

the circuit comes back in to the *ground state* at $t_2 \cong 2\pi / \Omega_R$



$$\omega_{10} = \frac{E_1 - E_0}{\hbar}$$

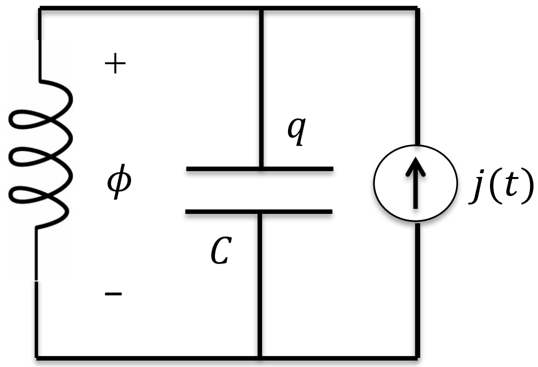
$$\Omega_R = \sqrt{(\omega - \omega_{10})^2 + \Omega_r^2}$$

3.2.1 Linear LC circuit: stationary states

Linear LC Circuit

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(\overbrace{-\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2}+\frac{1}{2L}\phi^2}^{\hat{E}}\right)\Psi \underbrace{- j(t)\phi\Psi}_{\text{source term}}$$

$$W_I = \frac{1}{2L}\phi^2$$



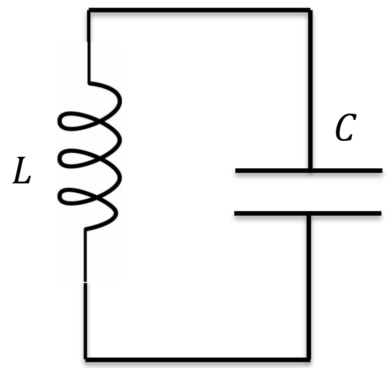
Energy Eigenstates

$$\hat{E} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2$$

$$\left(-\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2\right)\chi_n = E_n\,\chi_n,\ \ n = 0,1,2,\ldots$$

$$\omega_r = \frac{1}{\sqrt{LC}},\, Z_0 = \sqrt{\frac{L}{C}}$$

$$\boxed{\phi_c = \sqrt{\hbar Z_0}, q_c = \sqrt{\hbar/Z_0}}$$

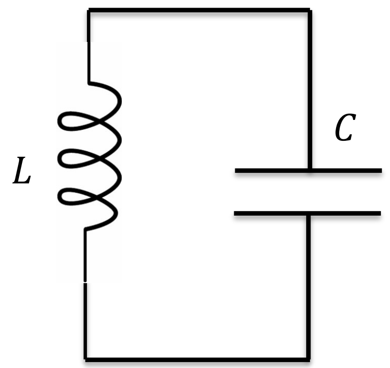


C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. I, 2nd ed., Wiley-VCH, 2019.

Energy Eigenstates

$$\hat{E} = -\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2$$

$$\left(-\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2\right)\chi_n = E_n\,\chi_n,\ \ n = 0,1,2,\ldots$$



$$\omega_r = \frac{1}{\sqrt{LC}},\ Z_0 = \sqrt{\frac{L}{C}}$$

$$\phi_c = \sqrt{\hbar Z_0},\ q_c = \sqrt{\hbar/Z_0}$$

$$\begin{cases} \hat{a} = \frac{1}{\sqrt{2}}\left(\frac{1}{\phi_c}\hat{\phi} + i\frac{1}{q_c}\hat{q}\right) \\ \hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\phi_c}\hat{\phi} - i\frac{1}{q_c}\hat{q}\right) \end{cases}$$

annihilation and
creation operators, they are not self-adjoint

$\hat{E} = \hbar\omega_r\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$ $\hat{N} = \hat{a}^\dagger\hat{a}$ is the number operator, it is self-adjoint $\hat{E} = \hbar\omega_r\left(\hat{N} + \frac{1}{2}\right)$

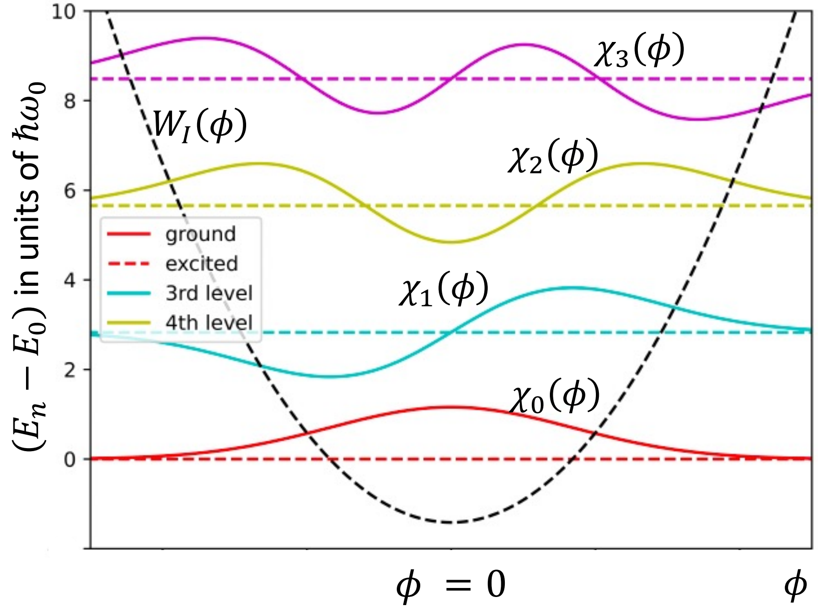
C. Cohen-Tannoudji, B. Diu and F. Laloë, Quantum Mechanics, vol. I, 2nd ed., Wiley-VCH, 2019.

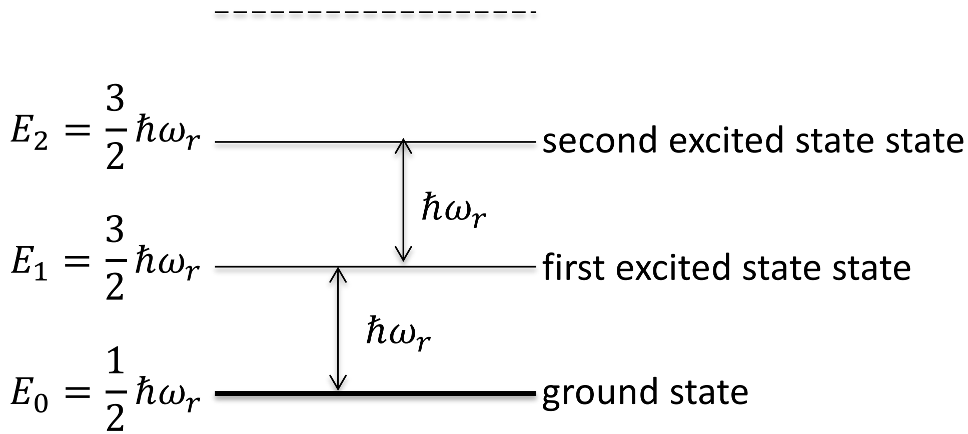
Energy Eigenstates

$$\hat{E} = \hbar\omega_r \left(\hat{N} + \frac{1}{2}\right), \hat{N} = \hat{a}^\dagger \hat{a}$$

$$\hat{N}\chi_n = n\chi_n \qquad n = 0,1,2,\ldots.$$

$$E_n = \hbar\omega_r \left(n + \frac{1}{2}\right) \text{ with } n = 0,1,2,\ldots$$





$\hbar\omega_r$ is the quantum of energy or the photon.

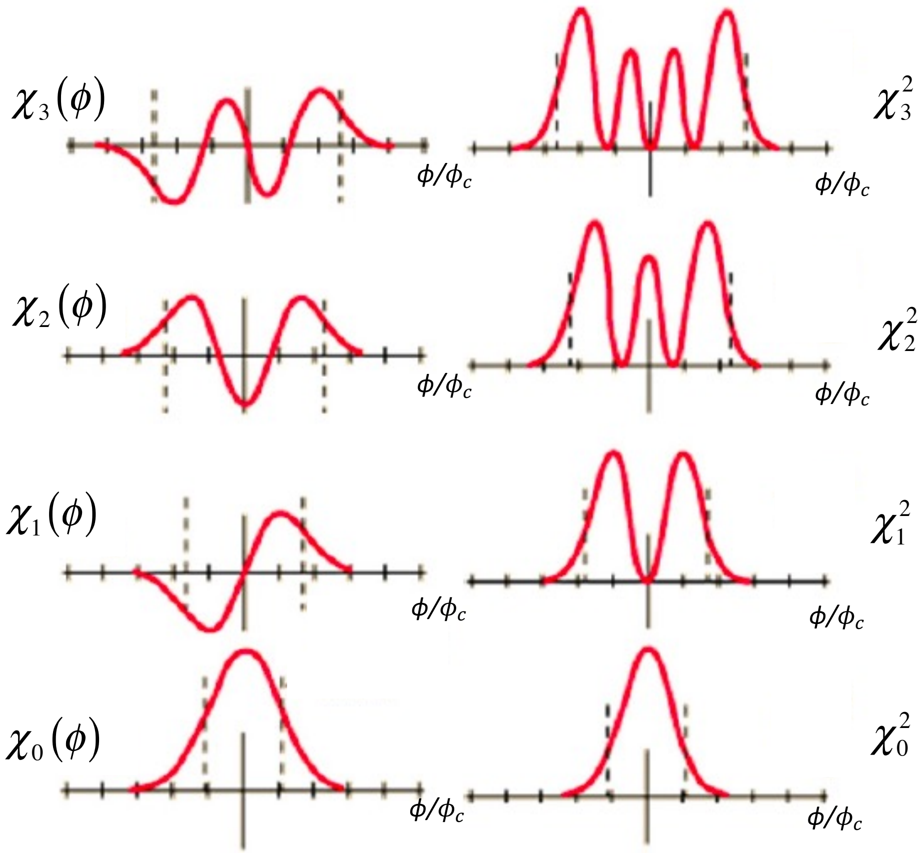
Energy Eigenstates of the linear LC Circuit

$E_3 = \frac{7}{2}\hbar\omega_r$

$E_2 = \frac{5}{2}\hbar\omega_r$

$E_1 = \frac{3}{2}\hbar\omega_r$

$E_0 = \frac{1}{2}\hbar\omega_r$



$$\langle \phi \rangle_{\chi_n} = \int \chi_n^*(\phi)\phi\chi_n(\phi)d\phi = 0$$

$$\langle q \rangle_{\chi_n} = \int \chi_n^*(\phi)\frac{\hbar}{i}\frac{d}{d\phi}\chi_n(\phi)d\phi = 0$$

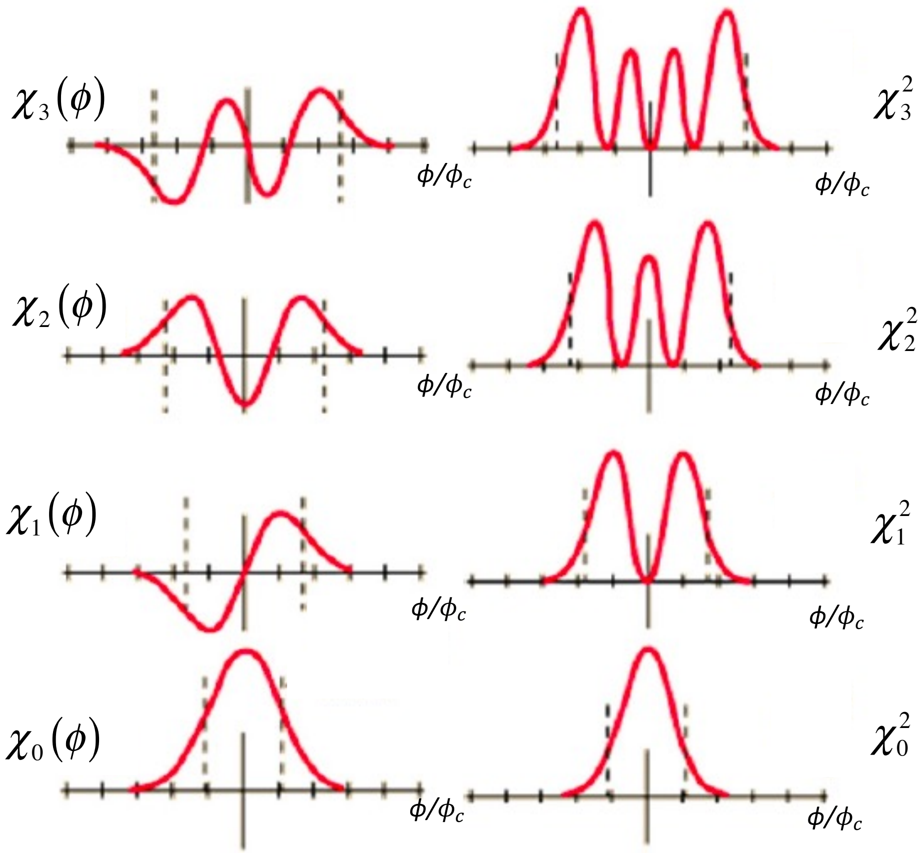
Energy Eigenstates of the linear LC Circuit

$$E_3 = \frac{7}{2}\hbar\omega_r$$

$$E_2 = \frac{5}{2}\hbar\omega_r$$

$$E_1 = \frac{3}{2}\hbar\omega_r$$

$$E_0 = \frac{1}{2}\hbar\omega_r$$



$$Z_0 = \sqrt{\frac{L}{C}}$$

$$\left\langle \phi^2 \right\rangle_{\chi_n} = \left(n + \frac{1}{2} \right) \hbar Z_0$$

$$\left\langle q^2 \right\rangle_{\chi_n} = \left(n + \frac{1}{2} \right) \frac{\hbar}{Z_0}$$

Fluctuations

$$\Phi_{zpf}^{(n)} = \sqrt{\langle \phi^2 \rangle_{\chi_n}},\ Q_{zpf}^{(n)} = \sqrt{\langle q^2 \rangle_{\chi_n}}$$ «fluctuations»

$$\Phi_{zpf}^{(n)} = \sqrt{\left(n+\frac{1}{2}\right)\hbar Z_0} = \sqrt{\left(n+\frac{1}{2}\right)}\phi_c,\ Q_{zpf}^{(n)} = \sqrt{\left(n+\frac{1}{2}\right)\frac{\hbar}{Z_0}} = \sqrt{\left(n+\frac{1}{2}\right)}q_c$$

$$\phi_c = \sqrt{\hbar Z_0}, q_c = \sqrt{\hbar / Z_0}$$

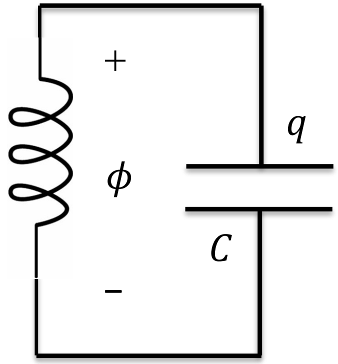
$\omega_r = 10^{11} \frac{rad}{s}, Z_0 = 10\Omega$ ⇒

- $\phi_c \cong 3.3 \times 10^{-17}$ V$\cdot s$, $\Phi_c \omega_r \cong 3.3\ \mu V$ voltage fuctuactions in the **ground state**
- $q_c \cong 3.3 \times 10^{-18}$ C, $Q_c \omega_r \cong 0.33\ \mu A$ current intensity fluctuations in the **ground state**

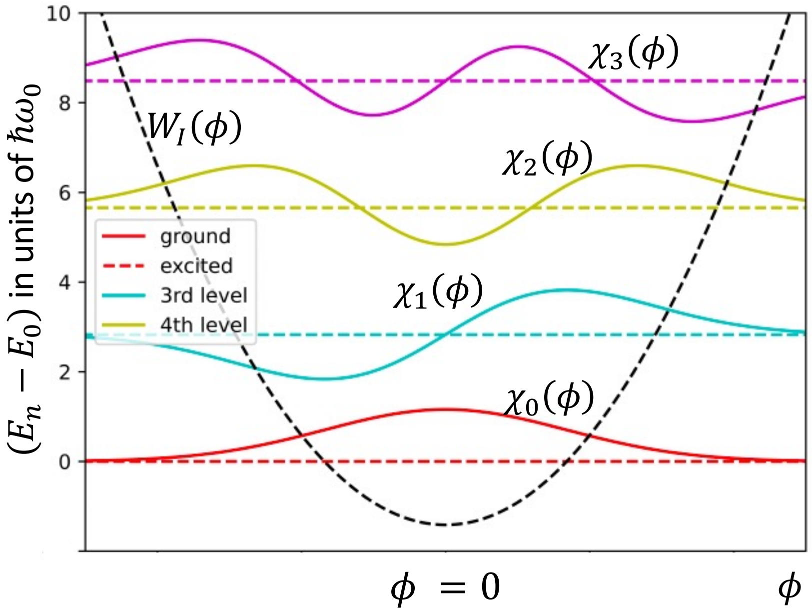
Stationary States

$$i\hbar \frac{\partial \Psi}{\partial t} = \overbrace{\left(-\frac{\hbar^2}{2C} \frac{\partial^2}{\partial \phi^2} + \frac{1}{2L} \phi^2 \right)}^{\hat{E}} \Psi$$

$$W_I = \frac{1}{2L} \phi^2$$



$E_n = \hbar \omega_r \left(n + \frac{1}{2} \right)$ with $n = 0,1,2, \ldots$ and $\omega_r = 1/\sqrt{LC}$

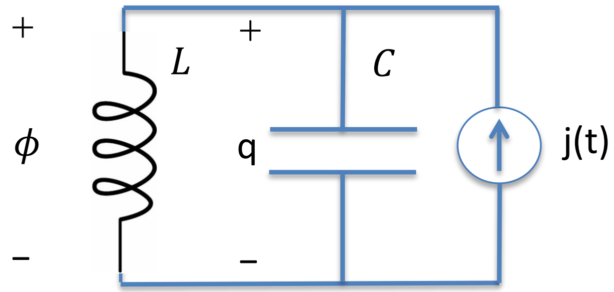


$$\Psi(\phi; t) = e^{-iE_n t/\hbar} \chi_n(\phi)$$

where

$$\hat{E}\, \chi_n(\phi) = E_n \chi_n(\phi)$$

Driven linear LC circuit



$$\overbrace{\quad}^{\hat{E}}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \underbrace{\left(-\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2\right)}_{}\Psi - \underbrace{j(t)\phi\Psi}_{\text{source term}}$$

$\hat{E}$ = $\left(-\frac{\hbar^2}{2C}\frac{\partial^2}{\partial\phi^2} + \frac{1}{2L}\phi^2\right)$

source term = $j(t)\phi\Psi$

Driven linear LC circuit

$$\begin{cases} i\hbar \dfrac{\partial \Psi}{\partial t} = \left(-\dfrac{\hbar^2}{2C}\dfrac{\partial^2}{\partial \phi^2} + \dfrac{1}{2L}\phi^2\right)\Psi - j(t)\phi\Psi \\ \Psi(\phi; t = 0) = \chi_0(\phi) \end{cases}$$

$$\Psi\,(\phi; t) = \sum_n c_n(t)\chi_n(\phi)e^{-i\omega_n t}$$

$$\dot{c}_m = \frac{1}{i\hbar}\sum_n W_{mn}(t)\, c_n(t)e^{i\omega_{mn}t} \text{ for } m = 0,1,2, \ldots,$$

where $\omega_{mn} = (E_m - E_n)/\hbar$, $W_{mn}(t) = j(t)\Phi_{mn}$ and $\Phi_{mn} = \sqrt{\frac{\hbar Z_0}{2}}\left(\sqrt{n+1}\delta_{m,n+1} + \sqrt{n}\delta_{m,n-1}\right)$. These equations are solved with the initial conditions $c_0(0) = 1$ and $c_n(0) = 0$ for $n = 1,2, \ldots$.

Driven linear LC circuit

$$\begin{cases} i\hbar \dfrac{\partial \Psi}{\partial t} = \left(-\dfrac{\hbar^2}{2C}\dfrac{\partial^2}{\partial \phi^2} + \dfrac{1}{2L}\phi^2\right)\Psi - j(t)\phi\Psi \\ \Psi(\phi; t=0) = \chi_0(\phi) \end{cases}$$

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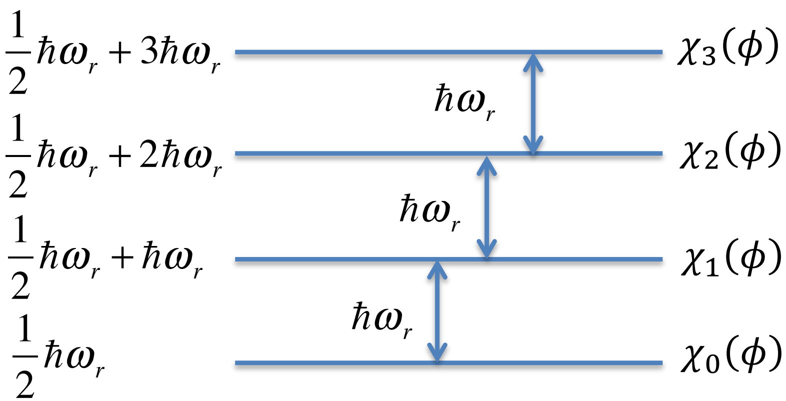
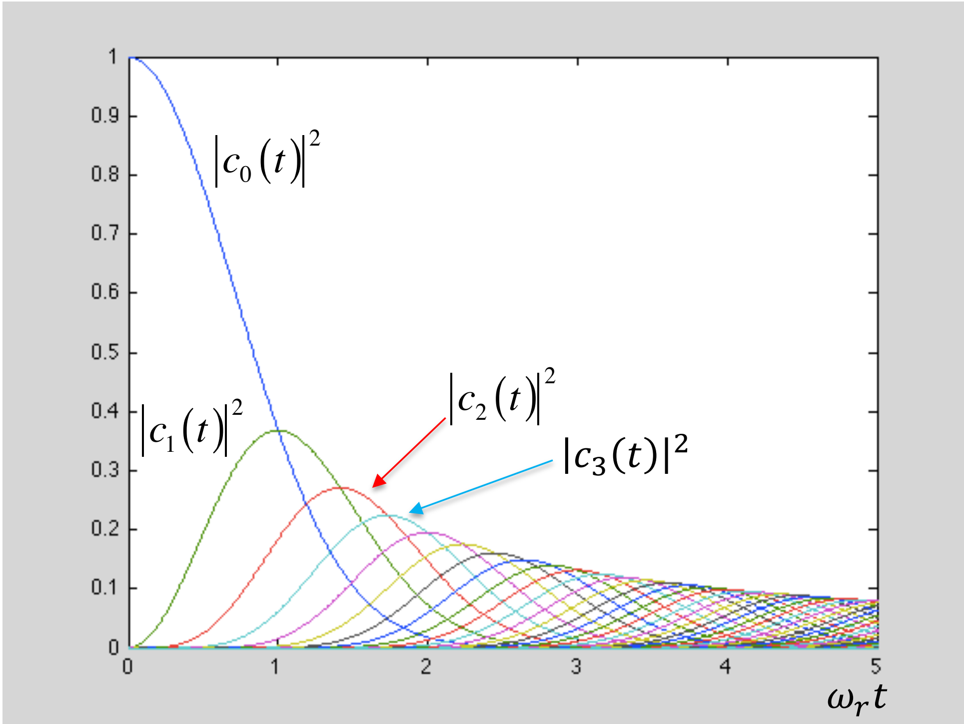


In this case the energy levels are uniformly spaced, therefore all the energy eigenstates can be resonantly excited. The linear LC circuit cannot behave as a two-level system.

Driven linear LC circuit

$j(t) = J_m sin(\omega t)$, with resonant condition $\omega \cong \omega_r$



Unlike the stationary states of an anharmonic oscillator all stationary states are excited (200 basis function).

3.2.2 Linear LC circuit: coherent quasi-classical states

Coherent Quasi – Classical state

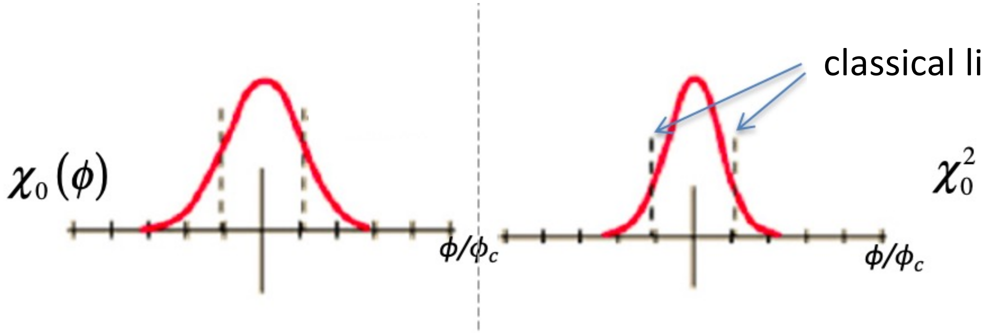
ϕ - representation

Stationary ground state

$$\Psi(\phi;t) = \chi_0(\phi) e^{-i\omega_r t/2} = \left(\frac{1}{\sqrt{\pi}\phi_c}\right)^{1/2} \exp\left(-\frac{\phi^2}{2\phi_c^2}\right) e^{-i\omega_r t/2}$$

classical limits $\frac{\phi_M}{\phi_c} = \sqrt{\frac{2E_0}{\hbar\omega_r}}$

$E_0 = \frac{1}{2}\hbar\omega_r$



Coherent Quasi – Classical state

ϕ - representation

Stationary ground state
$$\Psi(\phi;t)=\chi_0(\phi)e^{-i\omega_r t/2}=\left(\frac{1}{\sqrt{\pi}\phi_c}\right)^{1/2}\exp\left(-\frac{\phi^2}{2\phi_c^2}\right)e^{-i\omega_r t/2}$$

Coherent quasi-classical state
$$\Psi_{qcs}(\phi;t)=\left(\frac{1}{\sqrt{\pi\phi_c}}\right)^{1/2} exp\left\{-\frac{1}{2}\left[\frac{\phi-\langle\phi\rangle_{\Psi_{qsc}}(t)}{\phi_c}\right]^2\right\} exp\left\{i\left[\langle q\rangle_{\Psi_{qsc}}(t)\frac{\phi}{\hbar}-\frac{1}{2}\omega_r t\right]\right\}$$

where

$$\langle\phi\rangle_{\Psi_{qsc}}(t)=\int_{-\infty}^{+\infty}\Psi_{qsc}^*(\phi;t)\phi\,\Psi_{qsc}(\phi;t)d\phi$$ average flux

$$\langle q\rangle_{\Psi_{qsc}}(t)=\int_{-\infty}^{+\infty}\Psi_{qsc}^*(\phi;t)\frac{\hbar}{i}\frac{\partial}{\partial\phi}\Psi_{qsc}(\phi;t)d\phi$$ average charge

$$\Delta\phi=\frac{\phi_c}{\sqrt{2}},\Delta q=\frac{q_c}{\sqrt{2}}\implies\Delta\phi\Delta q=\frac{\hbar}{2}$$

Coherent Quasi – Classical State

ϕ - representation

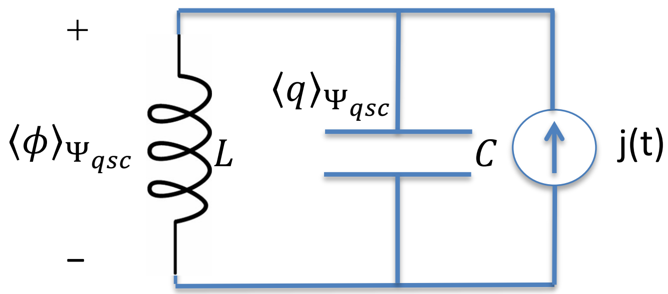
Stationary ground state $\Psi(\phi;t)=\chi_0(\phi)e^{-i\omega_r t/2}=\left(\frac{1}{\sqrt{\pi}\phi_c}\right)^{1/2}\exp\left(-\frac{\phi^2}{2\phi_c^2}\right)e^{-i\omega_r t/2}$

Coherent quasi-classical state $\Psi_{qcs}(\phi;t)=\left(\frac{1}{\sqrt{\pi\phi_c}}\right)^{1/2} exp\left\{-\frac{1}{2}\left[\frac{\phi-\langle\phi\rangle_{\Psi_{qsc}}(t)}{\phi_c}\right]^2\right\} exp\left\{i\left[\langle q\rangle_{\Psi_{qsc}}(t)\frac{\phi}{\hbar}-\frac{1}{2}\omega_r t\right]\right\}$

The function $\Psi_{qcs}(\phi;t)$ is solution of the Schröodinger equation provided that

Classical equations for the driven LC circuit

$$\begin{cases}\frac{d}{dt}\langle\phi\rangle_{\Psi_{qsc}}=\frac{1}{C}\langle q\rangle_{\Psi_{qsc}},\\ \frac{d}{dt}\langle q\rangle_{\Psi_{qsc}}=-\frac{1}{L}\langle\phi\rangle_{\Psi_{qsc}}+j(t),\\ \langle\phi\rangle_{\Psi_{qsc}}(t=0)=0,\langle q\rangle_{\Psi_{qsc}}(t=0)=0.\end{cases}$$



Quasi – classical state

ϕ - representation

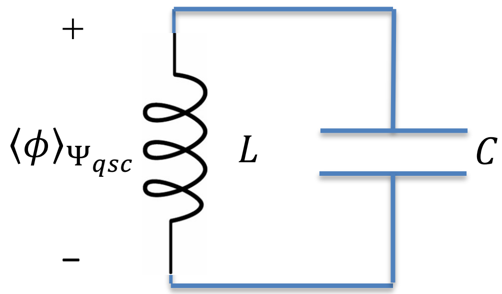
$$\Psi_{qcs}(\phi;t) = \left(\frac{1}{\sqrt{\pi\phi_c}}\right)^{1/2} exp\left\{-\frac{1}{2}\left[\frac{\phi - \langle\phi\rangle_{\Psi_{qsc}}(t)}{\phi_c}\right]^2\right\} exp\left\{i\left[\langle q\rangle_{\Psi_{qsc}}(t)\,\frac{\phi}{\hbar} - \frac{1}{2}\omega_r t\right]\right\}$$

Probability density

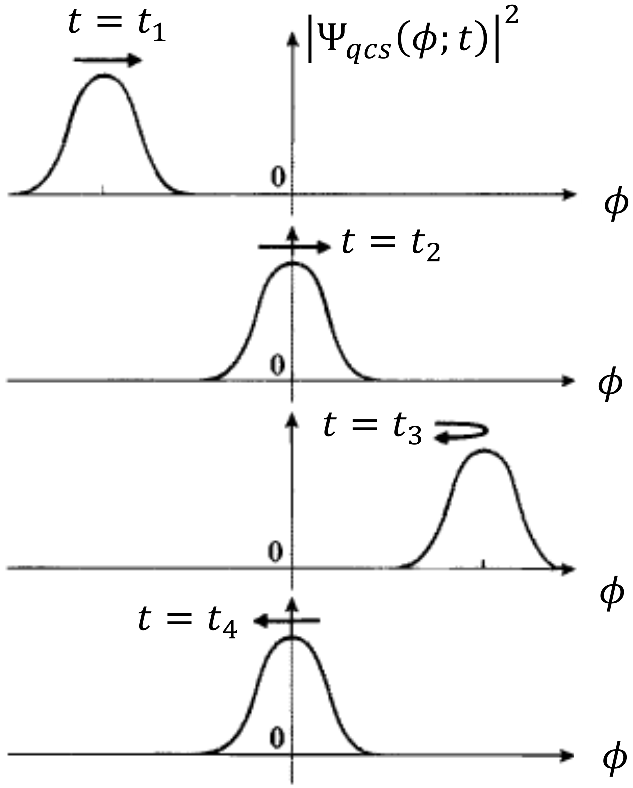
$$\left|\Psi_{qcs}(\phi;t)\right|^2 = \frac{1}{\sqrt{\pi\phi_c}} exp\left\{-\left[\frac{\phi - \langle\phi\rangle_{\Psi_{qsc}}(t)}{\phi_c}\right]^2\right\}$$

Time evolution of quasi-classical states: free evolution

$$\left|\Psi_{qcs}(\phi;t)\right|^2 = \frac{1}{\sqrt{\pi\phi_c}} exp\left\{-\left[\frac{\phi - \langle\phi\rangle_{\Psi_{qsc}}(t)}{\phi_c}\right]^2\right\}$$



$$\langle\phi\rangle_{\Psi_{qsc}}(t) = \Phi_m cos(\omega_r t + \gamma)$$



Time evolution of quasi-classical states: driven evolution

$$\alpha(t) \equiv \frac{1}{\sqrt{2}}\left[\frac{1}{\phi_c}\langle\phi\rangle_{\Psi_{qsc}} + i\frac{1}{q_c}\langle q\rangle_{\Psi_{qsc}}\right]$$

$$\langle E\rangle_{\Psi_{qsc}}(t) = \hbar\omega_r\left(|\alpha(t)|^2 + \frac{1}{2}\right)$$

where

$$\alpha(t) = \frac{i}{2q_c}\int_0^t e^{-i\omega_r(t-t')}j(t')dt'$$

and

$$|\alpha(t)|^2 = \frac{1}{\hbar\omega_r}\langle E\rangle_{\Psi_{qsc}}(t) - \frac{1}{2}$$

When $|\alpha(t)|^2 \gg 1$, $|\alpha(t)|^2$ gives the "average" number of photons stored in the circuit.

Photon generation in quasi – classical state

$$j(t) = J_m sin(\omega t)$$

$$\boxed{\alpha(t) = \frac{i}{2q_c}\int_0^t e^{-i\omega_r(t-t')} j(t')dt'}$$

If $\omega \approx \omega_r$ we obtain

$$\alpha(t) \cong \frac{i\Omega_0}{2\omega_-} e^{-i\omega_r t}\left(e^{i\omega_- t} - 1\right)$$

where $\omega_- = \omega_r - \omega$ and $\Omega_0 = \frac{J_m}{2q_c}$. Therefore, it results

$$|\alpha(t)|^2 \cong \left(\frac{\Omega_0}{\omega_-}\right)^2 sin^2\left(\frac{\omega_- t}{2}\right)$$

In the resonance condition, $\omega_r = \omega$ we obtain

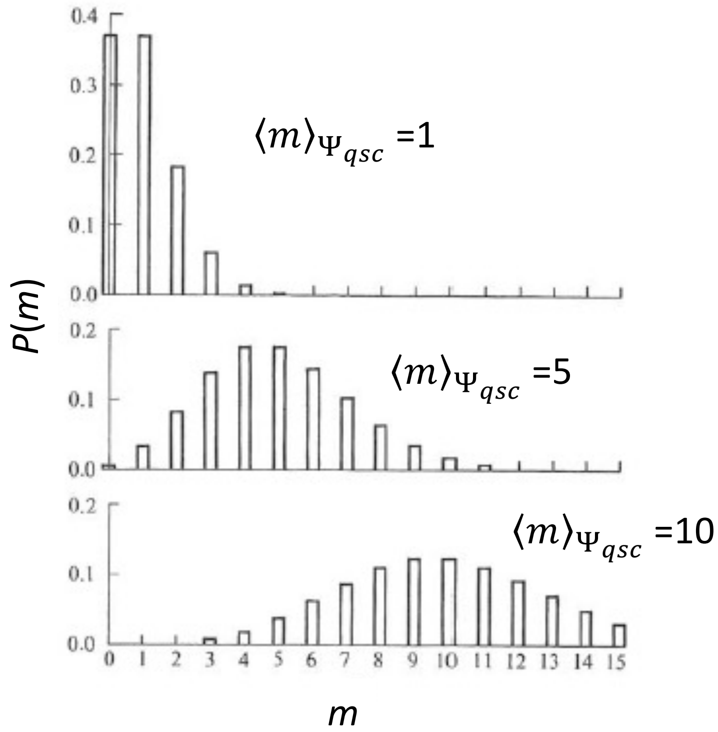
$$|\alpha(t)|^2 \cong \left(\frac{\Omega_0 t}{2}\right)^2$$

Photon generation in quasi – classical state

The <u>probability of finding the circuit in the energy eigenstate</u> $|\chi_m\rangle$ is given by the *Poisson distribution*

$$P_{\psi_{qsc}}(m;t) = \left|\langle\chi_m|\Psi_{qsc}\rangle\right|^2 = \frac{\left(\langle m\rangle_{\Psi_{qsc}}\right)^m}{m!} exp\left(-\langle m\rangle_{\Psi_{qsc}}\right) \text{ with } m = 0,1,2,\ldots$$

where $\langle m\rangle_{\Psi_{qsc}}$ is the <u>*average value of number of photons*</u> in the state $|\psi_{qsc}\rangle$: it <u>*varies in time and depends on the intensity of the classical current source*</u>.



Expectation value of photon number

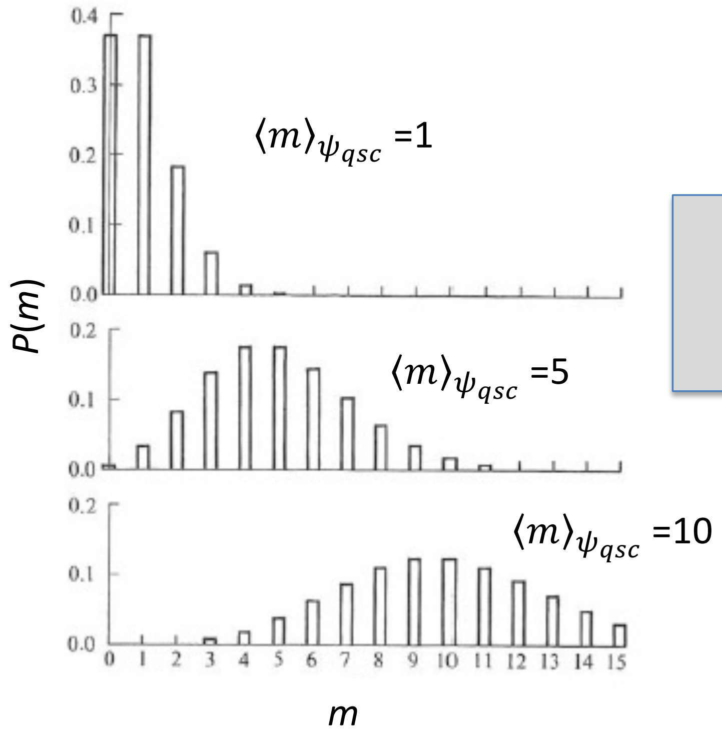
$$\langle m\rangle_{\Psi_{qsc}} = |\alpha(t)|^2$$

Photon generation in quasi – classical state

The probability of finding the circuit in the energy eigenstate $|\chi_m\rangle$ is given by the *Poisson distribution*

$$P_{\psi_{qsc}}(m;t) = \left|\langle \chi_m | \Psi_{qsc}\rangle\right|^2 = \frac{\left(\langle m\rangle_{\Psi_{qsc}}\right)^m}{m!} exp\left(-\langle m\rangle_{\psi_{qsc}}\right) \text{ with } m = 0,1,2,...$$

where $\langle m\rangle_{\psi_{qsc}}$ is the *average value of number of photons* in the state $|\psi_{qsc}\rangle$: it *varies in time and depends on the intensity of the classical current source*.



Expectation value of photon number

$$\langle m\rangle_{\Psi_{qsc}} = |\alpha(t)|^2$$

Variance of photon number

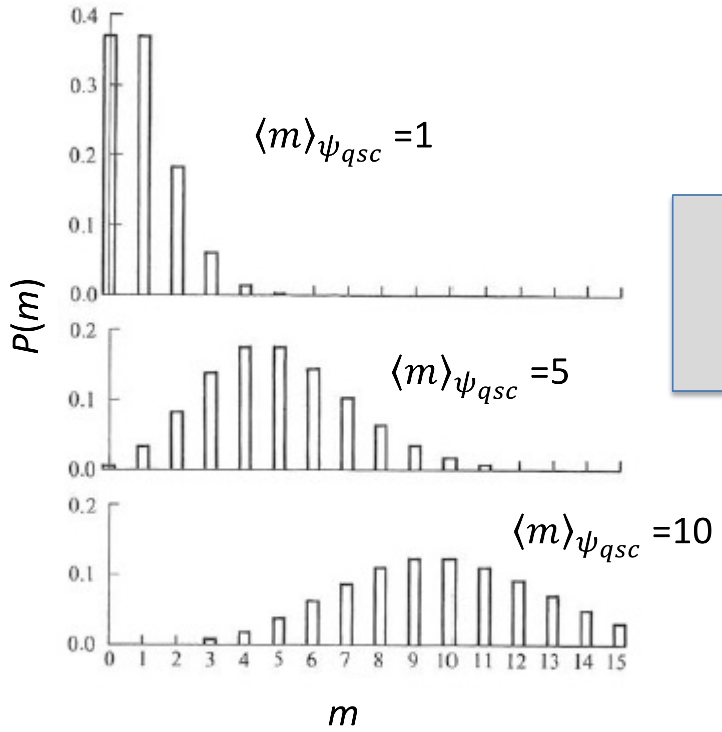
$$\langle \Delta m\rangle_{\Psi_{qsc}} = \sqrt{\langle m^2\rangle_{\Psi_{qsc}} - \langle m\rangle^2_{\Psi_{qsc}}} = |\alpha(t)|$$

Photon generation in quasi – classical state

The probability of finding the circuit in the energy eigenstate $|\chi_m\rangle$ is given by the *Poisson distribution*

$$P_{\Psi_{qsc}}(m;t) = \left|\langle\chi_m|\Psi_{qsc}\rangle\right|^2 = \frac{\left(\langle m\rangle_{\Psi_{qsc}}\right)^m}{m!} exp\left(-\langle m\rangle_{\Psi_{qsc}}\right) \text{ with } m = 0,1,2,\ldots$$

where $\langle m\rangle_{\psi_{qsc}}$ is the *average value of number of photons* in the state $|\psi_{qsc}\rangle$: it *varies in time and depends on the intensity of the classical current source*.



Expectation value of photon number

$$\langle m\rangle_{\Psi_{qsc}} = |\alpha(t)|^2$$

Variance of photon number

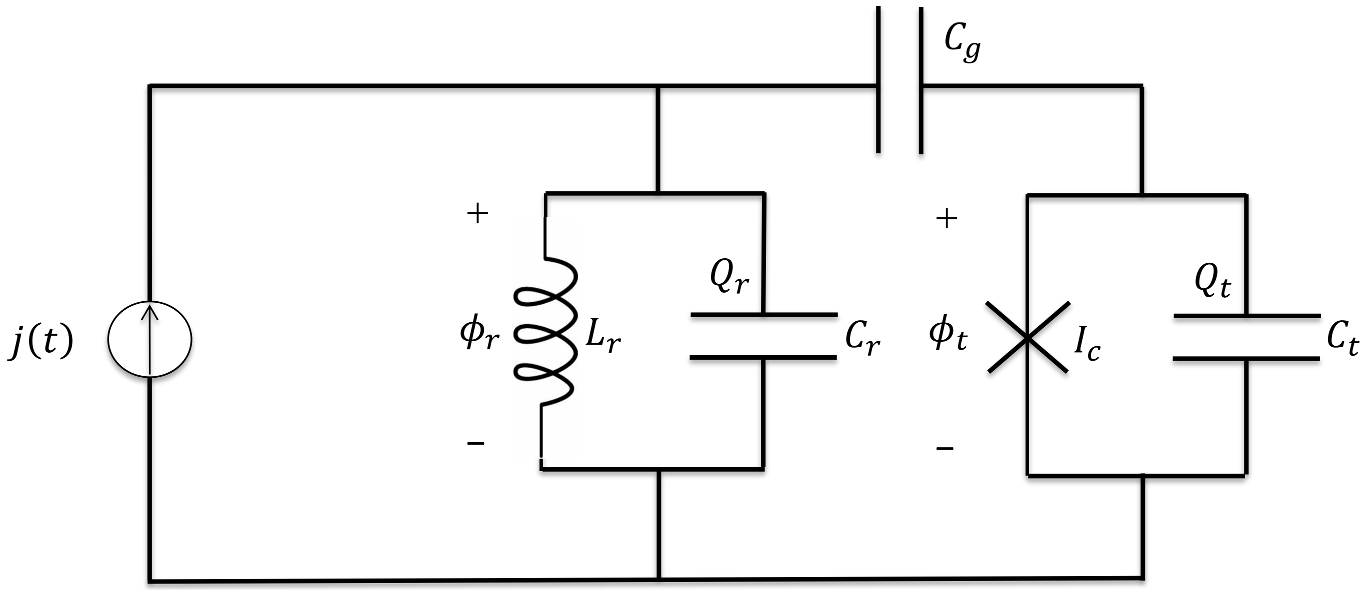
$$\langle\Delta m\rangle_{\Psi_{qsc}} = \sqrt{\langle m^2\rangle_{\Psi_{qsc}} - \langle m\rangle^2_{\Psi_{qsc}}} = |\alpha(t)|$$

$$\frac{\langle\Delta m\rangle_{\Psi_{qsc}}}{\langle m\rangle_{\Psi_{qsc}}} = \frac{1}{|\alpha|}$$

classical limit $|\alpha|$>>1

3.2.3 Linear LC circuit: Dispersive readout

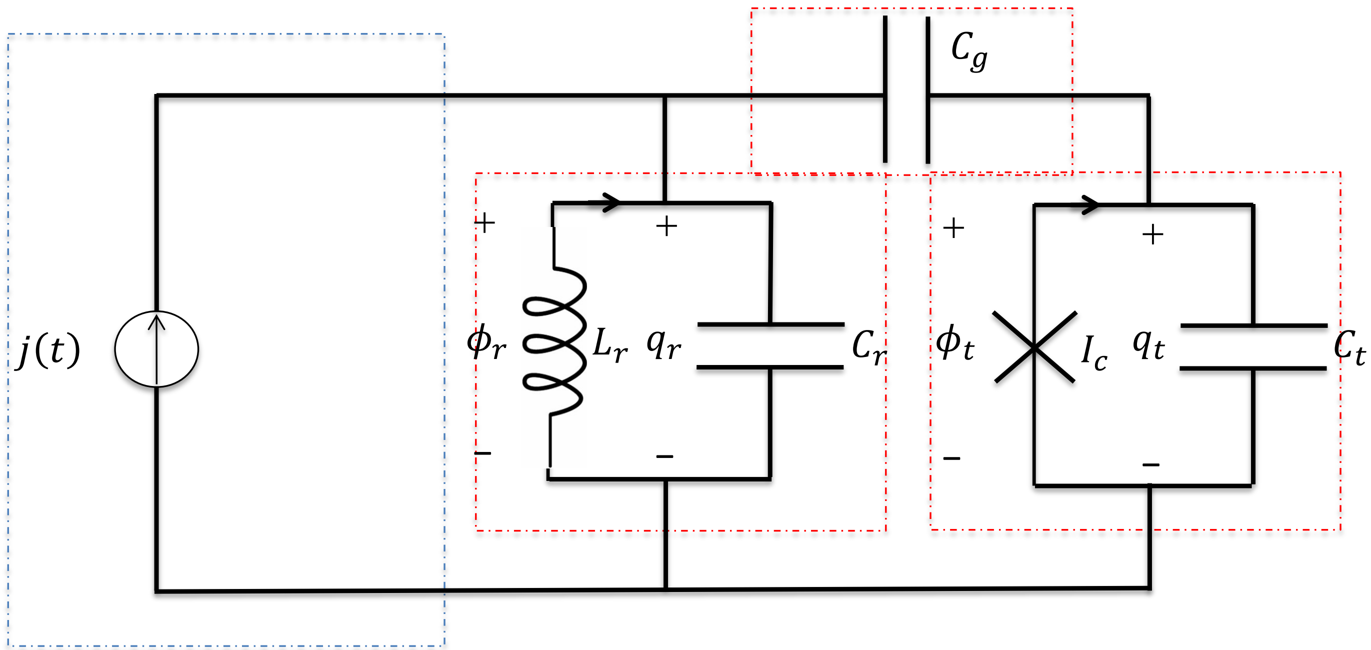
Transmon coupled to a resonator



This system has two degree of freedom: ϕ_r and ϕ_t.

In the weak coupling limit $C_r, C_t >> C_g$ Q_r is canonical conjugate to ϕ_r, and Q_t is canonical conjugate to ϕ_t.

Hamiltonian in the Weak Coupling



$$\underbrace{}_{}\hat{E}_r \quad \hat{E}_t \quad \hat{H}_{int} \quad \hat{H}_{drive}$$

$$H \cong \left(\frac{\hat{Q}_r^2}{2C_r} + \frac{\hat{\phi}_r^2}{2L_r}\right) + \left\{\frac{\hat{Q}_t^2}{2C_t} + E_J\left[1 - cos\left(\frac{2\pi\hat{\phi}_t}{\Phi_0}\right)\right]\right\} + \frac{C_g}{C_r C_t}\hat{Q}_r\,\hat{Q}_t - j(t)\phi_r$$

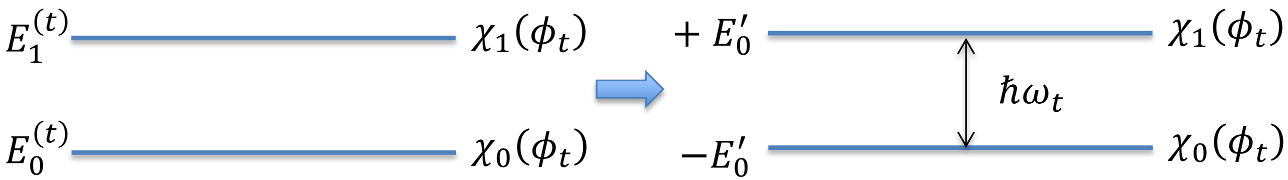
Effective Hamiltonian in the Dispersive Regime

- ω_t transition frequency of the transmon, $\omega_t = \frac{E_1^{(t)} - E_0^{(t)}}{\hbar}$
- Z_T characteristic impedance of the transmon, $Z_t = \sqrt{L_J/C_t}$ where $L_J = \frac{1}{2\pi}\frac{\Phi_0}{I_c}$

Effective Hamiltonian in the Dispersive Regime

- ω_t transition frequency of the transmon, $\omega_t = \frac{E_1^{(t)} - E_0^{(t)}}{\hbar}$
- Z_T characteristic impedance of the transmon, $Z_t = \sqrt{L_J/C_t}$ where $L_J = \frac{1}{2\pi}\frac{\Phi_0}{I_c}$

$$E_0' \equiv \hbar\omega_t/2$$



In the two-level approximation, the transmon Hamiltonian can be expressed as $\hat{E}_t = E_0'\hat{\sigma}$ where the Pauli operator $\hat{\sigma}$ is such that

$$\hat{\sigma}\chi_0(\phi_t) = -\chi_0(\phi_t) \text{ and } \hat{\sigma}\,\chi_1(\phi_t) = +\chi_1(\phi_t)$$

Effective Hamiltonian in the Dispersive Regime

- ω_t transition frequency of the transmon, $\omega_t = \frac{E_1^{(t)} - E_0^{(t)}}{\hbar}$
- Z_T characteristic impedance of the transmon, $Z_t = \sqrt{L_J/C_t}$ where $L_J = \frac{1}{2\pi}\frac{\Phi_0}{I_c}$

- ω_r natural frequency of the linear LC circuit, $\omega_r = 1/\sqrt{L_r C_r}$
- Z_r characteristic impedance of the linear LC circuit, $Z_r = \sqrt{L_r/C_r}$

$$\eta = \frac{\Omega^2}{\Delta\omega}, \Delta\omega = \omega_t - \omega_r, \Omega_c = \frac{1}{2}\frac{C_g}{C_r C_t}\frac{1}{\sqrt{Z_r Z_t}}$$

Effective Hamiltonian in the Dispersive Regime

- ω_t transition frequency of the transmon, $\omega_t = \frac{E_1^{(t)} - E_0^{(t)}}{\hbar}$
- Z_T characteristic impedance of the transmon, $Z_t = \sqrt{L_J/C_t}$ where $L_J = \frac{1}{2\pi}\frac{\Phi_0}{I_c}$

- ω_r natural frequency of the linear LC circuit, $\omega_r = 1/\sqrt{L_r C_r}$
- Z_r characteristic impedance of the linear LC circuit, $Z_r = \sqrt{L_r/C_r}$

$$\eta = \frac{\Omega^2}{\Delta\omega}, \ \Delta\omega = \omega_t - \omega_r, \quad \Omega_c = \frac{1}{2}\frac{C_g}{C_r C_t}\frac{1}{\sqrt{Z_r Z_t}}$$

Dispersive regime $\frac{\Omega_c}{|\Delta\omega|} \ll 1$

$$\hat{H}_{ef} \cong \frac{\hbar}{2}(\omega_t + \eta)\hat{\sigma} + \hbar(\omega_r + \eta\hat{\sigma})\hat{N} - j(t)\hat{\phi}$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$

A. Blais et al., Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation, PHYSICAL REVIEW A 69, 062320 (2004).

A. Blais et al., Circuit quantum electrodynamics, Reviews of Modern Physics 93, April-June 2021.

Dispersive coupling

We assume $\Delta\omega = \omega_t - \omega_r < 0$

$$\frac{\Omega_c}{|\Delta\omega|} \ll 1 \text{ dispersive regime} \quad \eta = \frac{\Omega^2}{\Delta\omega}$$

$$\underbrace{\hat{H}_{ef} \cong \frac{\hbar}{2}(\omega_t + \eta)\hat{\sigma}}_{\text{transmon}} + \underbrace{\hbar(\omega_r + \eta\hat{\sigma})\hat{N} - j(t)\hat{\phi}_r}_{\hat{H}'_r \text{ driven resonator}}$$

$$\boxed{\psi(\phi_t, \phi_r; t) \cong c_0(t)\xi_0(\phi_t)\Psi_0^{qsc}(\phi_r; t) + c_1(t)\xi_1(\phi_t)\Psi_1^{qsc}(\phi_r; t)}$$

Dispersive coupling

We assume $\Delta\omega = \omega_t - \omega_r < 0$

$$\frac{\Omega_c}{|\Delta\omega|} \ll 1 \text{ dispersive regime} \quad \eta = \frac{\Omega^2}{\Delta\omega}$$

$$\hat{H}_{ef} \cong \underbrace{\frac{\hbar}{2}(\omega_t + \eta)\hat{\sigma}}_{\text{transmon}} + \underbrace{\hbar(\omega_r + \eta\hat{\sigma})\hat{N} - j(t)\hat{\phi}_r}_{\hat{H}'_r \text{ driven resonator}}$$

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$\hat{\sigma}\chi_0(\phi_t) = -\chi_0(\phi_t)$ and $\hat{\sigma}\,\chi_1(\phi_t) = +\chi_1(\phi_t)$

$$\hat{H}'_r\psi(\phi_t, \phi_r; t) = c_0(t)\xi_0(\phi_t)\left[\hbar\left(\omega_r - \frac{\Omega^2}{\Delta\omega}\right)\hat{N} - j(t)\hat{\phi}_r\right]\Psi_0^{qsc}(\phi_r; t) + c_1(t)\xi_1(\phi_t)\left[\hbar\left(\omega_r + \frac{\Omega^2}{\Delta\omega}\right)\hat{N} - j(t)\hat{\phi}_r\right]\Psi_1^{qsc}(\phi_r; t)$$

$$\boxed{\left(\omega_r - \frac{\Omega^2}{\Delta\omega}\right) > \omega_r} \qquad \boxed{\left(\omega_r + \frac{\Omega^2}{\Delta\omega}\right) < \omega_r}$$

Dispersive coupling

We assume $\Delta\omega = \omega_t - \omega_r < 0$

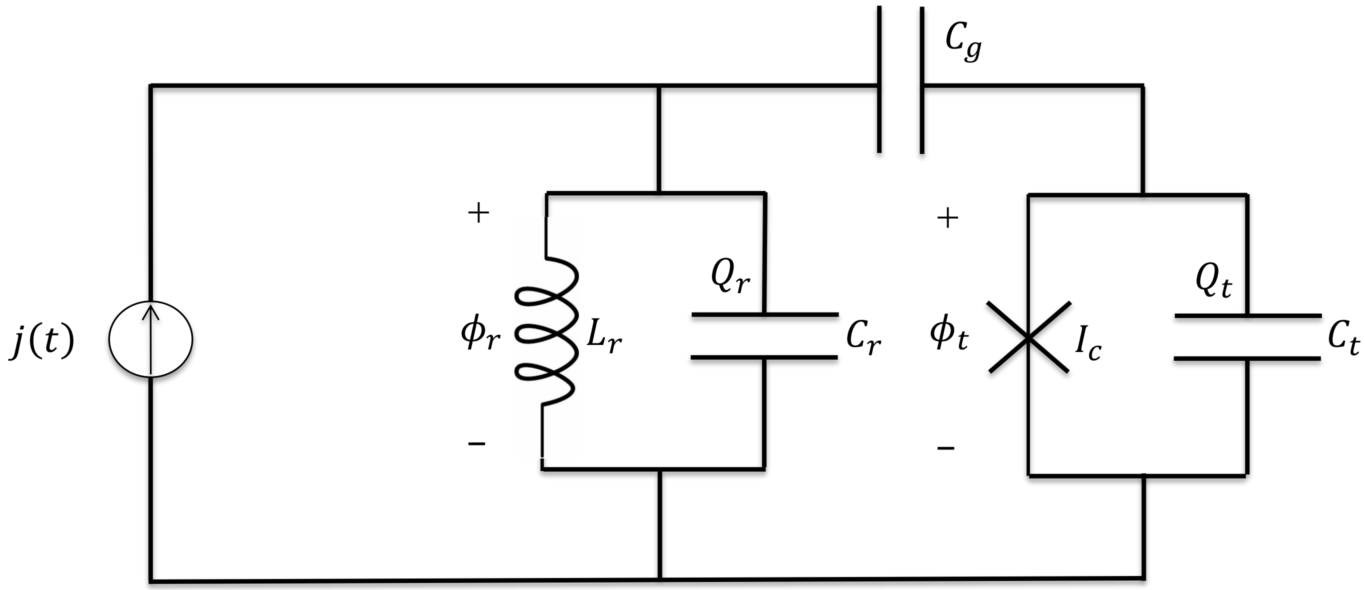
$$\frac{\Omega_c}{|\Delta\omega|} \ll 1 \text{ dispersive regime} \quad \eta = \frac{\Omega^2}{\Delta\omega}$$

$$\underbrace{\hat{H}_{ef} \cong \frac{\hbar}{2}(\omega_t + \eta)\hat{\sigma}}_{\text{transmon}} + \underbrace{\hbar\left(\omega_r + \frac{\Omega^2}{\Delta\omega}\hat{\sigma}\right)\hat{N} - j(t)\hat{\phi}_r}_{\text{driven resonator}}$$

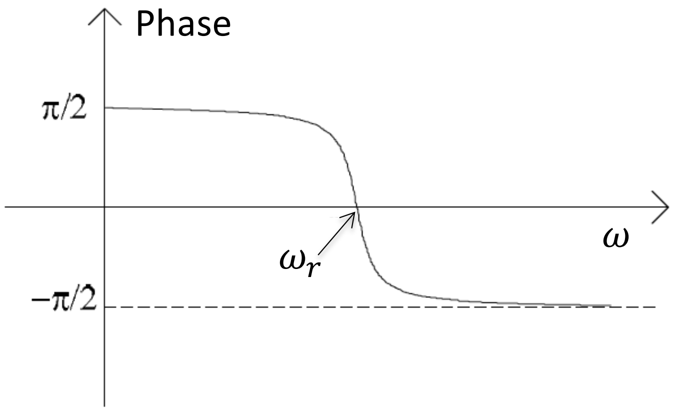
In the dispersive regime, the resonator frequency becomes qubit-state dependent:

- if the transmon is in the ground state $\chi_0(\phi_t)$ the resonator natural frequency is $\omega_r + \frac{\Omega_c^2}{|\Delta\omega|} > \omega_r$;
- if the transmon is in the excited state $\chi_1(\phi_t)$ the resonator natural frequency is $\omega_r - \frac{\Omega_c^2}{|\Delta\omega|} < \omega_r$.

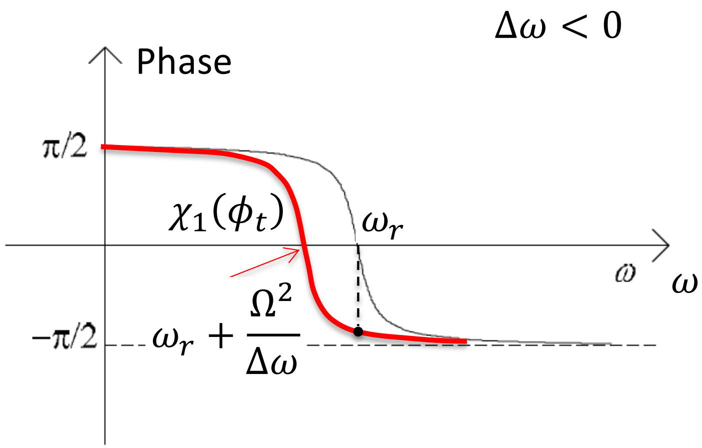
Dispersive qubit readout



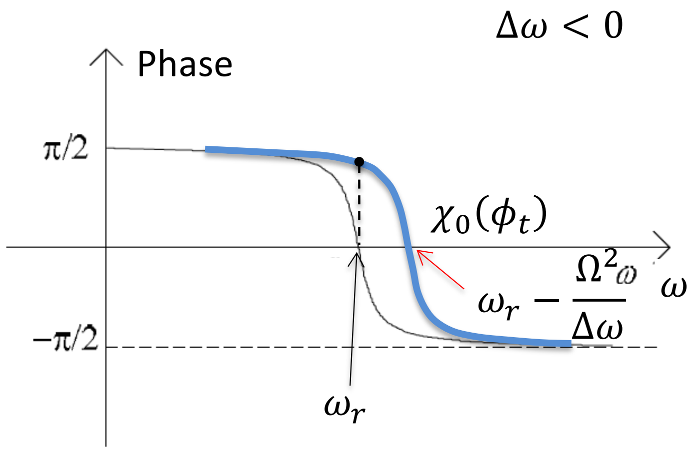
Phase response for the voltage $v_r = \dot{\phi}_r$ in absence of interaction with the transmon.



Dispersive qubit readout



phase response of LC circuit interacting with the transmon in the **excited state**: the phase at $\omega = \omega_r$ is **negative**.



phase response of LC circuit interacting with the transmon in the **ground state**: the phase at $\omega = \omega_r$ is **positive**.

Signal-to-Noise Ratio

Maximizing the Signal-to-Noise Ratio (SNR) is crucial. The SNR can be enhanced by increasing the probe power, i.e., the average number of photons $\langle N \rangle$, for the detection of the resonator state.

Nevertheless, $\langle N \rangle$ must be significantly less than the critical photon number

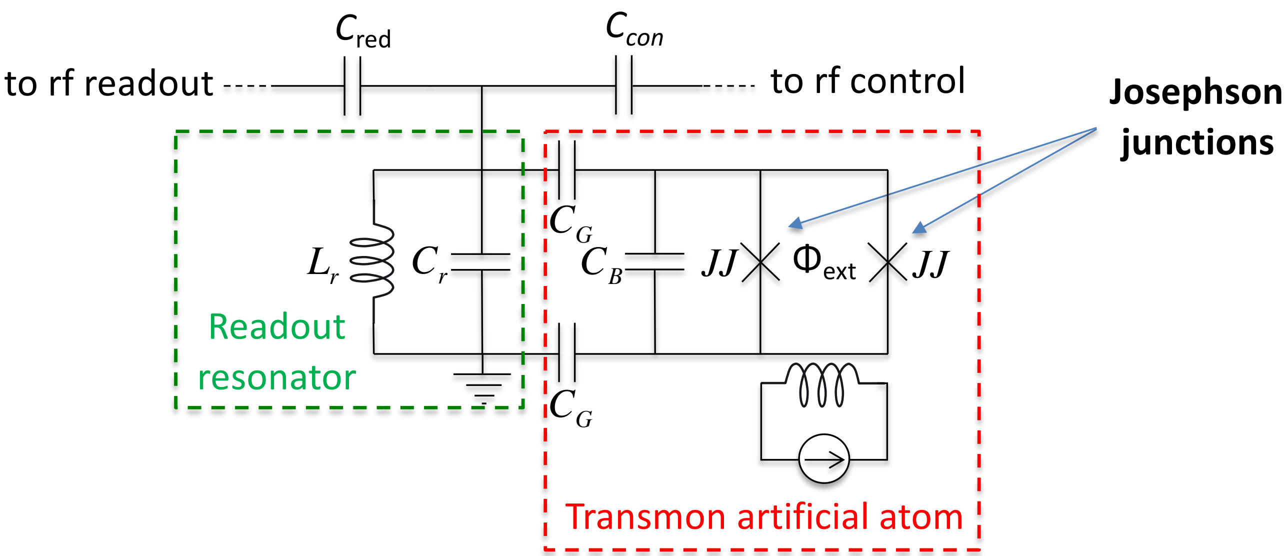
$$\langle N \rangle_c = (\Delta\omega/2\Omega_c)^2.$$

If not, the approximation

$$\hat{H}_{ef} \cong \frac{\hbar}{2}(\omega_t + \eta)\hat{\sigma}_z + \hbar(\omega_r + \eta\hat{\sigma}_z)\hat{N} - j(t)\hat{\phi}_r$$

is no longer valid. Then, photons induce unwanted qubit transitions to higher energy levels.

Superconducting Quantum Circuit for Quantum Computing



The transmon, which can implement a qubit, is realized through a nonlinear LC circuit.

The readout resonator, used to measure the transmon quantum state, consists of a linear LC circuit.

By firing coherent microwave signal, it is possible to control the qubit behavior and read its quantum state.

By courtesy of Alessandro Miano