Introduction to Circuit Quantum Electrodynamics

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Outline (1/2)

Outline (2/2)

3. SWAP gate
   - SWAP and iSWAP gates definition
   - Two coupled transmons: Hamiltonian in the *weak-coupling* regime
   - Two-level approximation
   - Circuit Hamiltonian in a two-level approximation
   - Schrödinger equation of a drive qubit and its solution in the rotating frame
   - Coupled Qubits in the Rotating Wave Approximation
   - Coordinates evolution and their matrix representation
   - Bell state generation

Classical bits

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- Each position in such a string is called a bit, and it contains either a 0 or a 1.

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- Each position in such a string is called a bit, and it contains either a 0 or a 1.
- To represent such collections of bits the computer must contain a corresponding **collection of physical systems**, each of which can exist in two unambiguously distinguishable physical states, associated with the value (0 or 1) of the abstract bit that the physical system represents.
- Such a physical system could be, for example:
  - a **switch** that could be open (0) or shut (1)
  - a **magnet** whose magnetization could be oriented in two different directions, up (0) or down (1)

Classical bits

We represent the state of each classical bit with a box, depicted by the symbol $|\cdot\rangle$, and denoted as **ket**, into which we place the value 0 or 1

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several equivalent notations are currently in use

$$\begin{array}{cccc} |0\rangle_2 & |1\rangle_2 & |2\rangle_2 & |3\rangle_2 \\ |0\rangle\,|0\rangle & |0\rangle\,|1\rangle & |1\rangle\,|0\rangle & |1\rangle\,|1\rangle \\ |0\rangle \otimes |0\rangle & |0\rangle \otimes |1\rangle & |1\rangle \otimes |0\rangle & |1\rangle \otimes |1\rangle \end{array}$$

Classical bits

The state of a pair of classical bits can be expressed as the tensor product of two classical bits

$$|x\rangle \otimes |y\rangle = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_0 y_0 \\ x_0 y_1 \\ x_1 y_0 \\ x_1 y_1 \end{pmatrix}$$

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$$|10\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \qquad |11\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix};$$

Reversible operations on one classical bit

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ERASE

$$\texttt{ERASE}: \begin{cases} |0\rangle \rightarrow |0\rangle \\ |1\rangle \rightarrow |0\rangle \end{cases}$$

`ERASE` is **irreversible**: given only the final state, there is no way to recover the initial state.

Reversible operations on one classical bit

Identity

$$\mathbf{1} = \begin{cases} |0\rangle \to |0\rangle \\ |1\rangle \to |1\rangle \end{cases}$$

- we express the identity operation by a linear operator $\mathbf{1}$ acting on the two-dimensional vector space
- The action of $\mathbf{1}$ on the column vectors $|0\rangle$, $|1\rangle$ is given by a matrix

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NOT (bit flip)

$$\mathtt{NOT} = \begin{cases} |0\rangle \to |1\rangle \\ |1\rangle \to |0\rangle \end{cases}$$

- The only non-trivial reversible operation we can apply to a single C-bit is the NOT
- NOT is reversible because it has an inverse: applying it a second time brings the state of the C-bit back to its original form.

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Reversible operation on multiple classical bits

A very common 2-Cbit operator is the tensor product $\otimes$ of two 1-Cbit operators

$$\mathbf{A} \otimes \mathbf{B} |xy\rangle = (\mathbf{A} \otimes \mathbf{B}) |x\rangle |y\rangle = (\mathbf{A} |x\rangle) \otimes (\mathbf{B} |y\rangle)$$

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$$\mathbf{A} \otimes \mathbf{B} \left| xy \right\rangle = (\mathbf{A} \otimes \mathbf{B}) \left| x \right\rangle \left| y \right\rangle = (\mathbf{A} \left| x \right\rangle) \otimes (\mathbf{B} \left| y \right\rangle)$$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}; \mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix}$$

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The matrix representation of the 2-Cbit operator $\mathbf{X} \otimes \mathbf{X}$ in the basis $|0\rangle_2, |1\rangle_2, |2\rangle_2, |3\rangle_2$ is

$$\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{X} \otimes \mathbf{X} |xy\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$\mathbf{X} \otimes \mathbf{X}$ operation

$$\mathbf{X} \otimes \mathbf{X} \left|00\right\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\left|11\right\rangle}; \quad \mathbf{S}_{10} \left|01\right\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\left|10\right\rangle}$$

$$\mathbf{X} \otimes \mathbf{X} \left|10\right\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{\left|01\right\rangle}; \quad \mathbf{X} \otimes \mathbf{X} \left|11\right\rangle = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{\left|00\right\rangle}$$

Swap Operation

One reversible operation we can apply to 2 C-bits is the SWAP operation

$$\mathbf{S}_{10} |xy\rangle \to |yx\rangle$$

$\|x\rangle$	$\|y\rangle$	$\mathbf{S}_{10} \|xy\rangle$
0	0	00
0	1	10
1	0	01
1	1	11

The matrix representation of the 2-Cbit operator is

$$\mathbf{S}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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Controlled C-not

reversible operation we can apply to two Cbits is the controlled-not or cNOT $\mathbf{C}_{ij}$

- State of the first C-bit (the control Cbit) is $|0\rangle$, then $\mathbf{C}_{10}$ leaves the state of the second Cbit (the target C-bit) unchanged
- if the state of the control C bit is $|1\rangle$, $\mathbf{C}_{10}$ applies the NOT operator to the state of the target C-bit.
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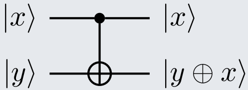
$$\mathbf{C}_{10}\,|x\rangle\,|y\rangle = |x\rangle\,|y \oplus x\rangle$$



$$\mathbf{C}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The modulo-2 sum $x \oplus y$ is also called the “exclusive OR” (or XOR)

Permutations

Permutations

The most general reversible operation on n C-bits in a classical computer is a permutation of the 2^n different basis states.

If $n = 2$, there are 4 basis states and $4!$ possible reversible operation

- The **states** that a classical bit can have are the two orthonormal vectors $|0\rangle$ and $|1\rangle$.

Qbits and their state

One Q-bit

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- The **state** $|\psi\rangle$ associated with a 1-Qbit can be any unit vector in the 2D vector space spanned by $|0\rangle$ and $|1\rangle$ over the complex numbers.

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$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle \qquad \text{with } |\alpha_0|^2 + |\alpha_1|^2 = 1$$

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$$|\psi\rangle = \alpha_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}$$

Qbits and their state

The state $|\psi\rangle$ is said to be a **superposition** of the states $|0\rangle$ and $|1\rangle$ with **amplitudes** α_0 and α_1.

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

$$0 < \theta < \pi; \; 0 < \varphi < 2\pi$$

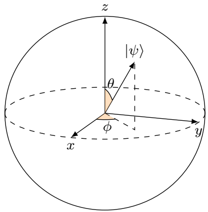

Figure: Bloch Sphere

The state Ψ associated with a 2-qbits quantum register is a normalized **superposition** of the four classical basis states

$$\Psi = \alpha_{00} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle$$

with the **normalization condition** $|\alpha_{01}|^2 + |\alpha_{01}|^2 + |\alpha_{01}|^2 + |\alpha_{01}|^2 = 1$

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Product States vs Entangled States

A **particular** 2-Qbit state Ψ is generated by the tensor product of two 1-Qbit states φ and ψ

$$\Psi = |\varphi\rangle \otimes |\psi\rangle = (a_0 |0\rangle + a_1 |1\rangle) \otimes (b_0 |0\rangle + b_1 |1\rangle) = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \begin{pmatrix} a_0 b_0 \\ a_0 b_1 \\ a_1 b_0 \\ a_1 b_1 \end{pmatrix}$$

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- A general two Qbit state is in the special form above if and only if $\alpha_{00}\alpha_{11} = \alpha_{10}\alpha_{10}$
- This state is denoted as **product state**

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Bell States

$$\Phi^{\pm} = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}} \qquad \Psi^{\pm} = \frac{|01\rangle \pm |01\rangle}{\sqrt{2}}$$

n Qbits register

This generalizes to n Qbits, whose state is a superposition of the 2^n different classical states,

$$|\Psi\rangle = \sum_{0 \leq x < 2^n} \alpha_x |x\rangle_n$$

with amplitudes whose squared magnitudes sum to unity $\sum_{0 \leq x < 2^n} |\alpha_x|^2 = 1$

Unitary transformation

- The only nontrivial reversible operation a classical computer can perform on a single C-bit is the NOT operation $\mathbb{X}$

Unitary operators

$$\mathbf{U}\mathbf{U}^{\dagger} = \mathbf{U}^{\dagger}\mathbf{U} = \mathbf{1}$$

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- The most general reversible operations that a quantum computer can perform upon a single Q-bit are represented by the action on the state of the Qbit of any **linear transformation** that takes **unit vectors into unit vectors**

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Unitary operators

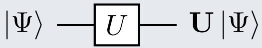
$$\mathbf{U}\mathbf{U}^\dagger = \mathbf{U}^\dagger\mathbf{U} = \mathbf{1}$$

- $\mathbf{U}$ is a **unitary** transformation, represented by a unitary matrix
- Since any unitary transformation has a unitary inverse, such actions of a quantum computer on a Qbit are **reversible**

Block diagrams

The action of a sequence of gates acting on n Qbits may be represented by block diagrams:

- Initially, the Qbit is described by the input state $|\Psi\rangle$.
- The thin line (wire) guides us through the subsequent history of the Qbit.



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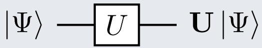
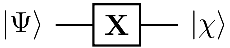
- After emerging from the box representing $\mathbf{U}$, the Qbit is described by the final state $\mathbf{U}\,|\Psi\rangle$.



Block diagrams

- Initially, the Qbits are described by the input state on the left.
- They are acted upon first by the gate $\mathbf{V}$ and then by the gate $\mathbf{U}$
- they emerging on the right in the final state $\mathbf{UV}\left|\Psi\right\rangle$.

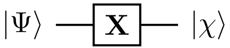
The order in which the Qbits encounter unitary gates in the figure is opposite to the order in which the corresponding symbols are written in the symbol for the final state on the right.



$$\mathbf{X} = \sigma_x : \begin{cases} |0\rangle \rightarrow |1\rangle \\ |1\rangle \rightarrow |0\rangle \end{cases}$$

Representation of the $\mathbf{X}$-gate in the basis $\{|0\rangle$

$$\mathbf{X} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

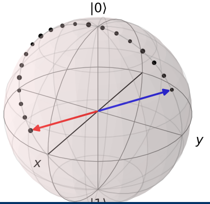


$$\mathbf{X} = \sigma_x : \begin{cases} |0\rangle \rightarrow |1\rangle \\ |1\rangle \rightarrow |0\rangle \end{cases}$$

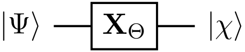
Representation of the $\mathbf{X}$-gate in the basis $\{|0\rangle$

$$\mathbf{X} = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In a Bloch sphere representation, the $\mathbf{X}$-gate performs a rotation of π around the x-axis



X_{Θ}-gate (arbitrary rotation around the x-axis)



Representation of the $\mathbf{X}_{\Theta}$ in the basis $\{|0\rangle$

$$\mathbf{X}_{\Theta} = \begin{vmatrix} -i\cos\left(\frac{\Theta_x}{2}\right) & \sin\left(\frac{\Theta_x}{2}\right) \\ \sin\left(\frac{\Theta_x}{2}\right) & -i\cos\left(\frac{\Theta_x}{2}\right) \end{vmatrix}$$

- In the Bloch sphere representation, the $\mathbf{X}_{\Theta}$-gate performs a rotation of Θ around the x-axis

X_Θ-gate (arbitrary rotation around the x-axis)

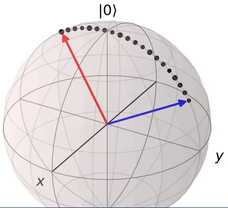
$$|\Psi\rangle \;—\boxed{\mathbf{X}_\Theta}—\; |\chi\rangle$$

Representation of the $\mathbf{X}_\Theta$ in the basis $\{|0\rangle$

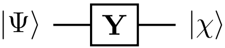
$$\mathbf{X}_\Theta = \begin{vmatrix} -i\cos\left(\frac{\Theta_x}{2}\right) & \sin\left(\frac{\Theta_x}{2}\right) \\ \sin\left(\frac{\Theta_x}{2}\right) & -i\cos\left(\frac{\Theta_x}{2}\right) \end{vmatrix}$$

- In the Bloch sphere representation, the $\mathbf{X}_\Theta$-gate performs a rotation of Θ around the x-axis
- For example, if $\Theta_x = \pi/2$ we have:

$$\mathbf{X}_{\frac{\pi}{2}} = \frac{1}{\sqrt{2}} \begin{vmatrix} -i & 1 \\ 1 & -i \end{vmatrix}$$



Y-gate



Matrix representation of the **Y**-gate in the basis $\{|0\rangle, |1\rangle\}$

$$\mathbf{Y} = \sigma_y : \begin{cases} |0\rangle \rightarrow i\,|1\rangle \\ |1\rangle \rightarrow -i\,|0\rangle \end{cases}$$

$$\mathbf{Y} = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Y-gate

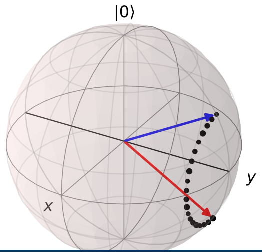
$$|\Psi\rangle \; -\boxed{\mathbf{Y}}- \; |\chi\rangle$$

Matrix representation of the $\mathbf{Y}$-gate in the basis $\{|0\rangle, |1\rangle\}$

$$\mathbf{Y} = \sigma_y : \begin{cases} |0\rangle \to i\,|1\rangle \\ |1\rangle \to -i\,|0\rangle \end{cases}$$

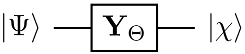
$$\mathbf{Y} = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

In a Bloch sphere representation, the $\mathbf{Y}$-gate performs a rotation of π around the y-axis



Y_Θ-gate (arbitrary rotation around the x-axis)

Representation of the $\mathbf{Y}_\Theta$-gate in the basis $\{|0\rangle, |1\rangle\}$



$$\mathbf{Y}_\Theta = \begin{vmatrix} -i\cos\left(\frac{\Theta_y}{2}\right) & \sin\left(\frac{\Theta_y}{2}\right) \\ \sin\left(\frac{\Theta_y}{2}\right) & -i\cos\left(\frac{\Theta_y}{2}\right) \end{vmatrix}$$

- In the Bloch sphere representation, the $\mathbf{Y}_\Theta$-gate performs a rotation of Θ_y around the y-axis

Y_Θ-gate (arbitrary rotation around the x-axis)

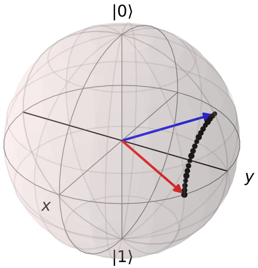
$$|\Psi\rangle \;—\boxed{\mathbf{Y}_\Theta}—\; |\chi\rangle$$

Representation of the $\mathbf{Y}_\Theta$-gate in the basis $\{|0\rangle, |1\rangle\}$

$$\mathbf{Y}_\Theta = \begin{vmatrix} -i\cos\left(\frac{\Theta_y}{2}\right) & \sin\left(\frac{\Theta_y}{2}\right) \\ \sin\left(\frac{\Theta_y}{2}\right) & -i\cos\left(\frac{\Theta_y}{2}\right) \end{vmatrix}$$

- In the Bloch sphere representation, the $\mathbf{Y}_\Theta$-gate performs a rotation of Θ_y around the y-axis
- For example, if $\Theta_y = \pi/2$ we have:

$$\mathbf{Y}_{\frac{\pi}{2}} = \frac{1}{\sqrt{2}}\begin{vmatrix} -i & 1 \\ 1 & -i \end{vmatrix}$$



$$|\Psi\rangle \;—\boxed{\mathbf{Z}}—\; |\chi\rangle$$

Matrix representation of the **Z**-gate in the basis $\{|0\rangle, |1\rangle\}$

$$\mathbf{Z} : \begin{cases} |0\rangle \to -|0\rangle \\ |1\rangle \to +|1\rangle \end{cases}$$

$$\mathbf{Z} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

Z-gate

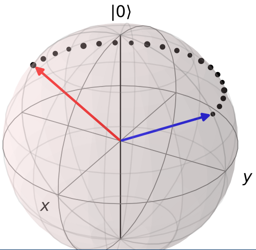
$$|\Psi\rangle \; -\boxed{\mathbf{Z}}- \; |\chi\rangle$$

$$\mathbf{Z} : \begin{cases} |0\rangle \to -|0\rangle \\ |1\rangle \to +|1\rangle \end{cases}$$

Matrix representation of the **Z**-gate in the basis $\{|0\rangle, |1\rangle\}$

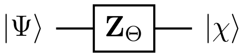
$$\mathbf{Z} = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

In a Bloch sphere representation, the **Z**-gate performs a rotation of π around the z-axis



Z_Θ-gate

arbitrary rotation around the z-axis

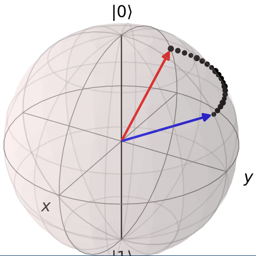


Representation of the $\mathbf{Z}_\Theta$-gate in the basis $\{|0\rangle, |1\rangle\}$

$$\mathbf{Z}_\Theta = \begin{vmatrix} -1 & 0 \\ 0 & -e^{-i\Theta} \end{vmatrix}$$

- In a Bloch sphere representation, the $\mathbf{Z}$-gate performs a rotation of π around the x-axis

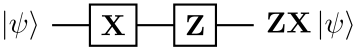
$$\mathbf{Z}_{\frac{\pi}{2}} = \begin{vmatrix} -1 & 0 \\ 0 & i \end{vmatrix}$$

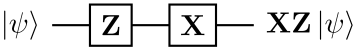


Gate Commutators

The order in which we apply the gates is very important (non-commutative)

$$[\mathbf{X}, \mathbf{Z}] \neq 0$$





$\mathbf{X}$ anticommutes with $\mathbf{Z}$

$$\mathbf{ZX} = -\mathbf{XZ}$$

Hadamard Gate

Definition

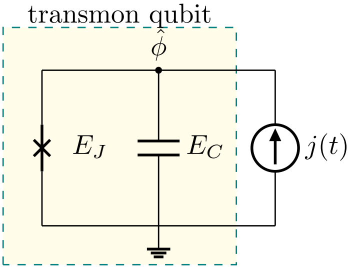
$$|\psi\rangle \; -\!\!\boxed{\mathbf{H}}\!\!- \; \mathbf{H}|\psi\rangle$$

$$\mathbf{H} = \frac{\mathbf{X} - \mathbf{Z}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

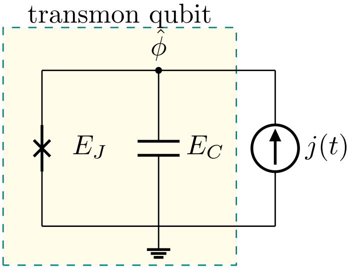
$$\mathbf{H}|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle\right)$$

$$\mathbf{H}|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \left(|0\rangle - |1\rangle\right)$$

Driven qubit



- One **degree of freedom**: flux $\hat{\phi}$;
- $\hat{q}$ is the **conjugate variable** to ϕ
- $\Phi_0 = 2\pi \frac{\hbar}{2e}$ is the flux quantum
- $E_C = \frac{(2e)^2}{2C}$
- E_J is the Josephson Energy



- One **degree of freedom**: flux $\hat{\phi}$;
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- $\Phi_0 = 2\pi \frac{\hbar}{2e}$ is the flux quantum
- $E_C = \frac{(2e)^2}{2C}$
- E_J is the Josephson Energy

$$\hat{H}(\hat{q}, \hat{\phi}; t) = \hat{H}_a + \hat{H}_{drive} = \left\{ \frac{\hat{q}^2}{2C} + E_J \left[1 - \cos\left(2\pi \frac{\hat{\phi}}{\Phi_0} \right) \right] \right\} - \mathrm{j}(t)\hat{\phi}$$

Driven qubit

$$j(t) = J_m f(t) \qquad J_m = \max_t \{j(t)\}$$

Driven qubit

$$j(t) = J_m f(t) \qquad J_m = \max_t \{j(t)\}$$

$$f(t) = s(t)\sin(\omega_d t + \gamma)$$

- $s(t)$ is the envelope function
- $\sin(\omega_d t + \gamma)$ is the high frequency carrier

Driven qubit

$$j(t) = J_m f(t) \qquad J_m = \max_t \{j(t)\}$$

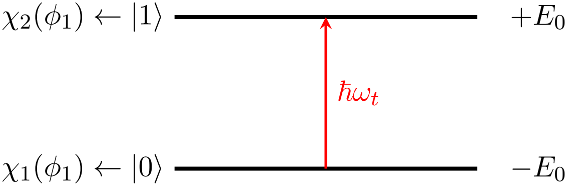
$$f(t) = s(t)\sin(\omega_d t + \gamma)$$

$$f(t) = s(t)\left[\underbrace{\cos(\gamma)}_{I}\sin(\omega_d t) + \underbrace{\sin(\gamma)}_{Q}\cos(\omega_d t)\right] = s(t)[I\sin(\omega_d t) + Q\cos(\omega_d t)]$$

- $s(t)$ is the envelope function
- $\sin(\omega_d t + \gamma)$ is the high frequency carrier
- $I = \cos\gamma$ is the in-phase component
- $Q = \sin\gamma$ is the out-of-phase component

Two-level approximated model: Transmon

- The transmon has **infinitely many** non-degenerate energy eigenstates
- its spectral lines are associated with transitions between any pair of energy eigenstates.



Two-level approximated model: Transmon

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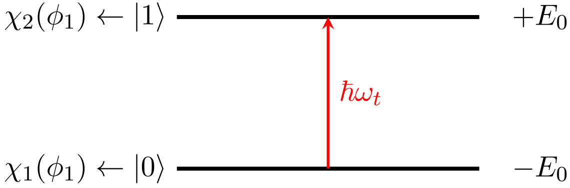
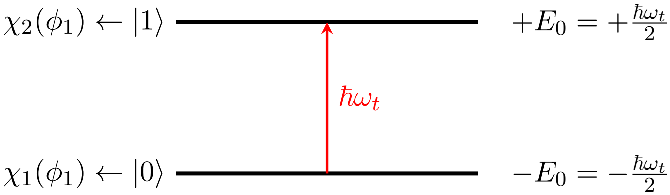


If the following hypotheses hold:

- large anharmonicity
- the frequency involved is such that only transitions between the two lowest levels are allowed.

The dynamics of the transmon qubit is obtained by the **two lowest energy levels**

Two-level approximated model: Transmon



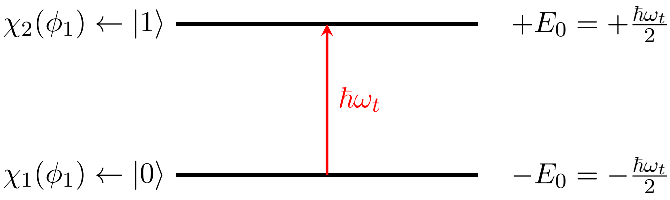
$$\omega_t = \sqrt{8E_C|E_J|} - \frac{E_C}{\hbar}$$

$$\widehat{H}_a \cong \frac{\hbar\omega_t}{2}\hat{\sigma}_z$$

$\hat{\sigma}_z$ definition

$$\begin{cases} \hat{\sigma}_z\,|0\rangle = -\,|0\rangle \\ \hat{\sigma}_z\,|1\rangle = +\,|1\rangle \end{cases}$$

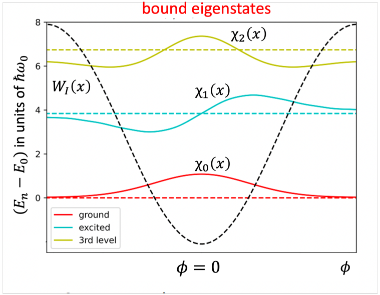
Two-level approximated model: Transmon



$$\hat{H}_{\text{drive}} = -j(t)\hat{\phi}$$

$$\widehat{H}_{drive} \cong -j(t)\phi_T\widehat{\sigma}_x$$

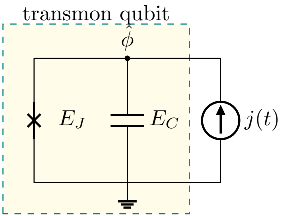
$$\phi_T \equiv 2\int_0^{\Phi_0/2} \chi_0(\phi)\phi\chi_1(\phi)d\phi$$



$\hat{\sigma}_x$ definition

$$\begin{cases} \hat{\sigma}_x |0\rangle = |1\rangle \\ \hat{\sigma}_x |1\rangle = |0\rangle \end{cases}$$

Driven qubit



- $\omega_t \approx \sqrt{8E_C|E_J|} - \dfrac{E_C}{\hbar}$
- $\phi_T \equiv 2\int_0^{\Phi_0/2} \chi_0(\phi)\phi\chi_1(\phi)d\phi$
- $j(t) = J_m f(t);$
- $f(t) = s(t)\sin(\omega_d t + \gamma); \quad J_m = \max_t\{s(t)\}$
- $\Omega = J_m \phi_T/\hbar$

two-level Hamiltonian

$$\hat{H} = \hbar\frac{\omega_t}{2}\hat{\sigma}_z + \hbar\Omega f(t)\hat{\sigma}_x$$

$\hat{\sigma}_z$ definition

$$\begin{cases} \hat{\sigma}_z\,|0\rangle = -\,|0\rangle \\ \hat{\sigma}_z\,|1\rangle = +\,|1\rangle \end{cases}$$

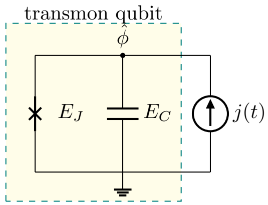
$\hat{\sigma}_z$ definition

$$\begin{cases} \hat{\sigma}_x\,|0\rangle = |1\rangle \\ \hat{\sigma}_x\,|1\rangle = |0\rangle \end{cases}$$

Qubit control & Logic Gates

The **state** of the circuit at time t is approx. described by the ket $|\psi(t)\rangle$ in a 2D Hilbert space

$$|\psi(t)\rangle = c_0(t)\,|0\rangle + c_1(t)\,|1\rangle$$



- $|c_0(t)|^2$ yields the probability that the outcome of the measurement of energy of the qubit at time t is $-E_0$
- $|c_1(t)|^2$ yields the probability that at time t the outcome of the measurement of energy of the qubit at time t is $+E_0$

$$|\psi(0)\rangle = c_0(0)\,|0\rangle + c_1(0)\,|1\rangle$$

Qubit control & Logic Gates

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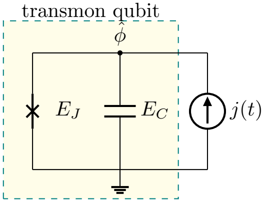


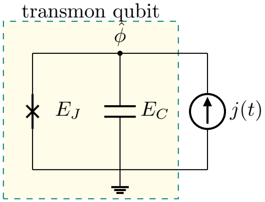
- $|c_0(t)|^2$ yields the probability that the outcome of the measurement of energy of the qubit at time t is $-E_0$
- $|c_1(t)|^2$ yields the probability that at time t the outcome of the measurement of energy of the qubit at time t is $+E_0$
- At $t = 0$ the state of the quantum circuit is $|\psi(0)\rangle = |\psi_0\rangle$

$$|\psi(0)\rangle = c_0(0)\,|0\rangle + c_1(0)\,|1\rangle$$

Qubit control & Logic Gates

The **state** of the circuit at time t is approx. described by the ket $|\psi(t)\rangle$ in a 2D Hilbert space

$$|\psi(t)\rangle = c_0(t)\,|0\rangle + c_1(t)\,|1\rangle$$



- To find the state of the quantum circuit $|\psi(t)\rangle$ at time t, we have to solve the Schrödinger equation

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = \left[\hbar\frac{\omega_t}{2}\hat{\sigma}_z - \hbar\Omega f(t)\hat{\sigma}_x\right]|\psi(t)\rangle$$
$$|\psi\left(t_0\right)\rangle = |\psi_0\rangle$$

Variable transformation: rotating frame

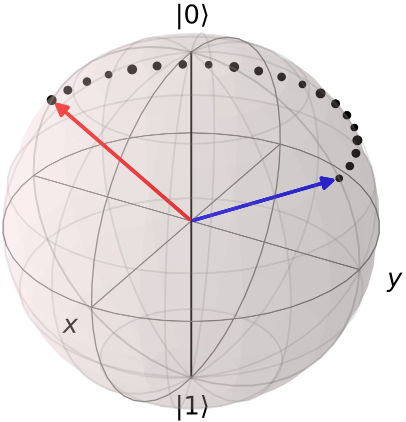
$$i\frac{d|\psi(t)\rangle}{dt} = \left[\frac{\omega_t}{2}\hat{\sigma}_z - \Omega f(t)\hat{\sigma}_x\right]|\psi(t)\rangle$$

- We solve the problem in a new variable $|\psi'\rangle$.

Variable transformation: rotating frame

$$i\frac{d|\psi(t)\rangle}{dt} = \left[\frac{\omega_t}{2}\hat{\sigma}_z - \Omega f(t)\hat{\sigma}_x\right]|\psi(t)\rangle$$

- We solve the problem in a new variable $|\psi'\rangle$.
- At time t, the Bloch sphere representation of $|\psi'\rangle$ is rotated clockwise about the z-axis by an angle $\alpha(t) = +\omega_d t$ with respect to the Bloch sphere representation of $|\psi\rangle$



Variable transformation: rotating frame

$$i\frac{d|\psi(t)\rangle}{dt} = \left[\frac{\omega_t}{2}\hat{\sigma}_z - \Omega f(t)\hat{\sigma}_x\right]|\psi(t)\rangle$$

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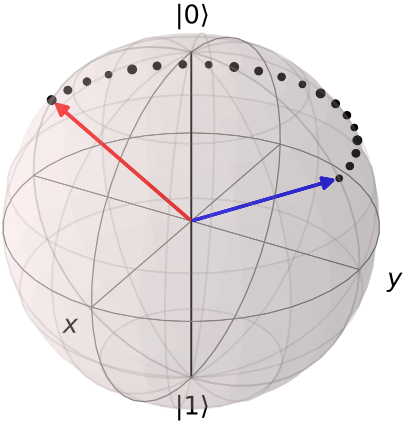
- At time t, the Bloch sphere representation of $|\psi'\rangle$ is rotated clockwise about the z-axis by an angle $\alpha(t) = +\omega_d t$ with respect to the Bloch sphere representation of $|\psi\rangle$
- $\widehat{R}_z(\alpha(t)) = -e^{i\alpha/2}\mathbf{Z}_\alpha$



$$|\psi(t)\rangle = \widehat{R}_z(\alpha(t))\,|\psi'(t)\rangle \quad \text{where} \quad \widehat{R}_z(\alpha(t)) = \exp\left(-i\alpha(t)\frac{\hat{\sigma}_z}{2}\right) \quad \alpha(t) = -\omega_d t$$

Variable transformation: rotating frame

$$i\frac{d|\psi(t)\rangle}{dt} = \left[\frac{\omega_t}{2}\hat{\sigma}_z - \Omega f(t)\hat{\sigma}_x\right]|\psi(t)\rangle$$

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Variable transformation: rotating frame

$$i\frac{d|\psi(t)\rangle}{dt} = \left[\frac{\omega_t}{2}\hat{\sigma}_z - \Omega f(t)\hat{\sigma}_x\right]|\psi(t)\rangle$$

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- At time t, the Bloch sphere representation of $|\psi'\rangle$ is rotated clockwise about the z-axis by an angle $\alpha(t) = +\omega_d t$ with respect to the Bloch sphere representation of $|\psi\rangle$

$$i\frac{d\,|\psi'\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}'_z - \Omega f(t)\hat{\sigma}'_x\right]|\psi'\rangle$$

We have to find the expression of the operators $\hat{\sigma}'_z$ $\hat{\sigma}'_x$ in the rotating frame.

Variable transformation: rotating frame

$$i\frac{d\,|\psi'\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}'_z - \Omega f(t)\hat{\sigma}'_x\right]|\psi'\rangle$$

$$\begin{cases} \hat{\sigma}'_z = \hat{R}_z^\dagger \hat{\sigma}_z \hat{R}_z \\ \hat{\sigma}'_x = \hat{R}_z^\dagger \hat{\sigma}_x \hat{R}_z \end{cases}$$

$$\begin{cases} \hat{\sigma}'_z = \hat{R}_z^\dagger \hat{\sigma}_z \hat{R}_z = \hat{\sigma}_z \\ \hat{\sigma}'_x(t) = \hat{R}_z^\dagger \hat{\sigma}_x \hat{R}_z = \hat{\sigma}_x \cos\left(\omega_d t\right) + \hat{\sigma}_y \sin\left(\omega_d t\right) \end{cases}$$

Variable transformation: rotating frame

$$i\frac{d\left|\psi'\right\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}'_z - \Omega f(t)\hat{\sigma}'_x\right]\left|\psi'\right\rangle$$

$$\begin{cases} \hat{\sigma}'_z = \hat{R}_z^\dagger \hat{\sigma}_z \hat{R}_z \\ \hat{\sigma}'_x = \hat{R}_z^\dagger \hat{\sigma}_x \hat{R}_z \end{cases}$$

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Schrödinger equation in the rotating frame

$$i\frac{d\left|\psi'\right\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}_z - \Omega f(t)\left(\hat{\sigma}_x \cos\left(\omega_d t\right) + \hat{\sigma}_y \sin\left(\omega_d t\right)\right)\right]\left|\psi'\right\rangle$$

Variable transformation: rotating frame

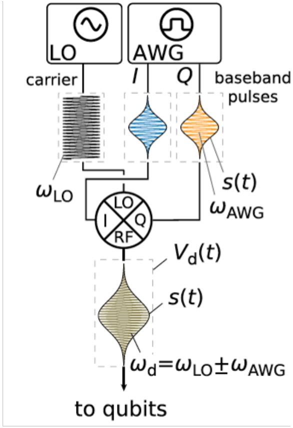
$$i\frac{d\left|\psi'\right\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}_z + \Omega f(t)\left(\hat{\sigma}_x \cos\left(\omega_d t\right) - \hat{\sigma}_y \sin\left(\omega_d t\right)\right)\right]\left|\psi'\right\rangle$$

$$f(t) = s(t)\sin(\omega_d t + \gamma) = s(t)\left[\underbrace{\cos(\gamma)}_{I}\sin(\omega_d t) + \underbrace{\sin(\gamma)}_{Q}\cos(\omega_d t)\right]$$

- $s(t)$ is the envelope function
- $I = \cos\gamma$ is the in-phase component; $Q = \sin\gamma$ is the out-of-phase component

$$i\frac{d\left|\psi'\right\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}_z + \Omega s(t)\left[I\sin(\omega_d t) + Q\cos(\omega_d t)\right]\left(\hat{\sigma}_x \cos\left(\omega_d t\right) - \hat{\sigma}_y \sin\left(\omega_d t\right)\right)\right]\left|\psi'\right\rangle$$

- **Local Oscillator (LO).** This generates a **high-frequency carrier** signal at frequency ω_{LO}. It acts as the reference signal for the mixing.



Control signals

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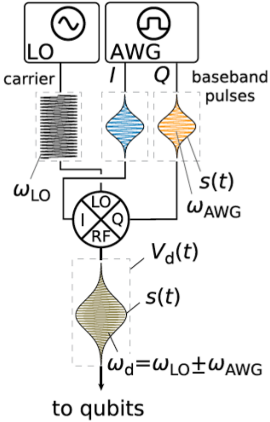
- **Arbitrary Waveform Generator (AWG).** This produces baseband signals (I and Q components) that represent the desired control pulses. These are lower-frequency signals typically shaped for specific qubit operations (e.g., Gaussian or shaped pulses).



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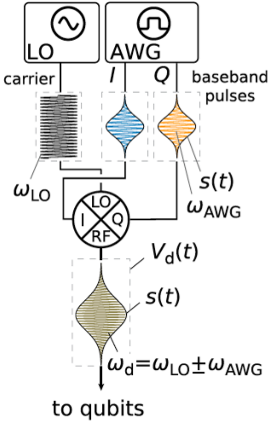

Control signals

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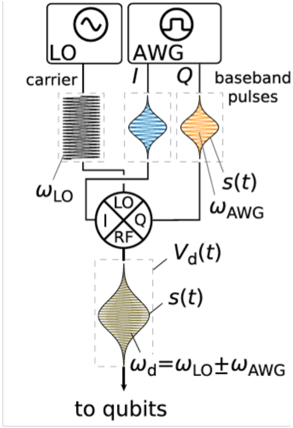
- **Baseband Pulses (I and Q)**: These are the output signals from the AWG. The I (In-phase) and Q (Quadrature) components define the amplitude and phase of the control pulses.
- **IQ Mixer**. Combines the carrier signal from the LO with the baseband signals I and Q. This produces a modulated signal, $V_d(t)$, with a frequency $\omega_d = \omega_{\text{LO}} \pm \omega_{\text{AWG}}$. This mixing process shifts the baseband signal to the desired frequency range for driving the qubit.



Schrödinger equation in the rotating frame

$$i\frac{d\left|\psi'\right\rangle}{dt}=\left[\frac{\Delta\omega}{2}\hat{\sigma}_z-\Omega s(t)\left[I\sin(\omega_d t)+Q\cos(\omega_d t)\right]\left(\hat{\sigma}_x\cos\left(\omega_d t\right)-\hat{\sigma}_y\sin\left(\omega_d t\right)\right)\right]\left|\psi'\right\rangle$$

Schrödinger equation in the rotating frame

$$i\frac{d\left|\psi'\right\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}_z - \Omega s(t)\left[I\sin(\omega_d t) + Q\cos(\omega_d t)\right]\left(\hat{\sigma}_x\cos\left(\omega_d t\right) - \hat{\sigma}_y\sin\left(\omega_d t\right)\right)\right]\left|\psi'\right\rangle$$

$$s(t)\left[I\sin\left(\omega_d t\right) + Q\cos\left(\omega_d t\right)\right]\left[\hat{\sigma}_x\cos\left(\omega_d t\right) + \hat{\sigma}_y\sin\left(\omega_d t\right)\right] =$$

Schrödinger equation in the rotating frame

$$i\frac{d|\psi'\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}_z - \Omega s(t)\left[I\sin(\omega_d t) + Q\cos(\omega_d t)\right]\left(\hat{\sigma}_x\cos(\omega_d t) - \hat{\sigma}_y\sin(\omega_d t)\right)\right]|\psi'\rangle$$

$$s(t)\left[I\sin(\omega_d t) + Q\cos(\omega_d t)\right]\left[\hat{\sigma}_x\cos(\omega_d t) + \hat{\sigma}_y\sin(\omega_d t)\right] =$$

$$s(t)\left[I\hat{\sigma}_x\sin(\omega_d t)\cos(\omega_d t) + I\hat{\sigma}_y\sin^2(\omega_d t) + Q\hat{\sigma}_x\cos^2(\omega_d t) + Q\hat{\sigma}_y\sin(\omega_d t)\cos(\omega_d t)\right] =$$

Schrödinger equation in the rotating frame

$$i\frac{d\,|\psi'\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}_z - \Omega s(t)\left[I\sin(\omega_d t) + Q\cos(\omega_d t)\right]\left(\hat{\sigma}_x\cos\left(\omega_d t\right) - \hat{\sigma}_y\sin\left(\omega_d t\right)\right)\right]|\psi'\rangle$$

$$s(t)\left[I\sin\left(\omega_d t\right) + Q\cos\left(\omega_d t\right)\right]\left[\hat{\sigma}_x\cos\left(\omega_d t\right) + \hat{\sigma}_y\sin\left(\omega_d t\right)\right] =$$

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$$\frac{s(t)}{2}\left[I\hat{\sigma}_x\sin\left(2\omega_d t\right) + I\hat{\sigma}_y\left[1-\cos\left(2\omega_d t\right)\right] + Q\hat{\sigma}_x\left[1+\cos\left(2\omega_d t\right)\right] + Q\hat{\sigma}_y\sin\left(2\omega_d t\right)\right] =$$

Schrödinger equation in the rotating frame

$$i\frac{d\left|\psi'\right\rangle}{dt} = \left[\frac{\Delta\omega}{2}\hat{\sigma}_z - \Omega s(t)\left[I\sin(\omega_d t) + Q\cos(\omega_d t)\right]\left(\hat{\sigma}_x\cos\left(\omega_d t\right) - \hat{\sigma}_y\sin\left(\omega_d t\right)\right)\right]\left|\psi'\right\rangle$$

$$s(t)\left[I\sin\left(\omega_d t\right) + Q\cos\left(\omega_d t\right)\right]\left[\hat{\sigma}_x\cos\left(\omega_d t\right) + \hat{\sigma}_y\sin\left(\omega_d t\right)\right] =$$

$$s(t)\left[I\hat{\sigma}_x\sin\left(\omega_d t\right)\cos\left(\omega_d t\right) + I\hat{\sigma}_y\sin^2\left(\omega_d t\right) + Q\hat{\sigma}_x\cos^2\left(\omega_d t\right) + Q\hat{\sigma}_y\sin\left(\omega_d t\right)\cos\left(\omega_d t\right)\right] =$$

$$\frac{s(t)}{2}\left[I\hat{\sigma}_x\sin\left(2\omega_d t\right) + I\hat{\sigma}_y\left[1-\cos\left(2\omega_d t\right)\right] + Q\hat{\sigma}_x\left[1+\cos\left(2\omega_d t\right)\right] + Q\hat{\sigma}_y\sin\left(2\omega_d t\right)\right] =$$

$$\frac{s(t)}{2}\left[I\hat{\sigma}_y + Q\hat{\sigma}_x\right] + \frac{s(t)}{2}\left\{\underbrace{\left[I\hat{\sigma}_x + Q\hat{\sigma}_y\right]}\sin\left(2\omega_d t\right) + \underbrace{\left[Q\hat{\sigma}_x - I\hat{\sigma}_y\right]}\cos\left(2\omega_d t\right)\right\}$$

Schrödinger equation in the rotating frame

$$i\hbar\frac{d\,|\psi'\rangle}{dt} = \left[i\hbar\frac{\Delta\omega}{2}\hat{\sigma}_z \underbrace{-i\hbar\Omega\frac{s(t)}{2}\left[I\hat{\sigma}_y + Q\hat{\sigma}_x\right]}_{H_{drive}^{\mathtt{RWA}}} \underbrace{-i\hbar\Omega\frac{s(t)}{2}\left[\hat{A}\sin(2\omega_d t) + \hat{B}\cos(2\omega_d t)\right]}_{H_{drive}^{(2\omega_d)}} \right] |\psi'\rangle$$

$$\begin{cases} H_{drive}^{(\mathtt{RWA})} = -\hbar\Omega\frac{s(t)}{2}\left[I\hat{\sigma}_y + Q\hat{\sigma}_x\right] \\ H_{drive}^{(2\omega_d)} = -\hbar\Omega\frac{s(t)}{2}\left\{\hat{A}\sin(2\omega_d t) + \hat{B}\cos(2\omega_d t)\right\} \end{cases}$$

$$i\hbar\frac{d\,|\psi'\rangle}{dt} = \left[\hbar\frac{\Delta\omega}{2}\hat{\sigma}_z + H_{drive}^{(\mathtt{RWA})} + H_{drive}^{(2\omega_d)}\right]|\psi'\rangle$$

Rotating Wave Approximation

$$i\hbar \frac{d\left|\psi'\right\rangle}{dt} = \left[\hbar\frac{\Delta\omega}{2}\hat{\sigma}_z + H_{drive}^{(\mathtt{RWA})} + \cancel{H_{drive}^{(2\omega_d)}}\right]\left|\psi'\right\rangle \approx \left[\hbar\frac{\Delta\omega}{2}\hat{\sigma}_z + H_{drive}^{(\mathtt{RWA})}\right]\left|\psi'\right\rangle$$

- When $\Omega \ll \omega_d$, the rapidly oscillating terms in the driving Hamiltonian, denoted as $\hat{H}_{\text{drive}}^{(2\omega_d)}$, oscillate at frequency $2\omega_d$ and can be neglected.
- These terms **average out to zero** over time, as their contribution is negligible on the longer time scales dictated by $\Delta\omega(t)$ and $s(t)$.
- This simplification is known as the Rotating Wave Approximation (RWA), which retains only the terms responsible for the dominant system dynamics.

Z-control gate

$$\begin{cases} i\dfrac{d\left|\psi'\right\rangle}{dt} = \left[\dfrac{\Delta\omega(t)}{2}\hat{\sigma}_z - s(t)\dfrac{\Omega}{2}\left(Q\hat{\sigma}_x + I\hat{\sigma}_y\right)\right]\left|\psi'\right\rangle \\ \left|\psi'\left(0\right)\right\rangle = \left|\psi_0\right\rangle \end{cases}$$

Z-control gate

$$\begin{cases} i\dfrac{d\left|\psi'\right\rangle}{dt} = \left[\dfrac{\Delta\omega(t)}{2}\hat{\sigma}_z - s(t)\dfrac{\Omega}{2}\left(Q\hat{\sigma}_x + I\hat{\sigma}_y\right)\right]\left|\psi'\right\rangle \\ \left|\psi'(0)\right\rangle = \left|\psi_0\right\rangle \end{cases}$$

- We **switch-off** the driving, setting $s(t) = 0$

$$\begin{cases} i\dfrac{d\left|\psi'\right\rangle}{dt} = \dfrac{\Delta\omega(t)}{2}\hat{\sigma}_z\left|\psi'\right\rangle \\ \left|\psi'(0)\right\rangle = \left|\psi_0\right\rangle \end{cases} \qquad \text{with} \quad \Delta\omega(t) = \omega_t(t) - \omega_d$$

Z-control gate

$$\begin{cases} i\dfrac{d\left|\psi'\right\rangle}{dt} = \left[\dfrac{\Delta\omega(t)}{2}\hat{\sigma}_z - s(t)\dfrac{\Omega}{2}\left(Q\hat{\sigma}_x + I\hat{\sigma}_y\right)\right]\left|\psi'\right\rangle \\ \left|\psi'(0)\right\rangle = \left|\psi_0\right\rangle \end{cases}$$

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- After integration, we arrive at the following solution:

$$\left|\psi'(t)\right\rangle = \exp\left[-i\frac{\Theta_z(t)t}{2}\hat{\sigma}_z\right]\left|\psi_0\right\rangle \quad \text{with} \quad \Theta_z(t) = \int_0^t \Delta\omega(\tau)d\tau$$

Z-control gate

$$\left|\psi'(t)\right\rangle = \exp\left[-i\frac{\Theta_z(t)t}{2}\hat{\sigma}_z\right]\left|\psi_0\right\rangle \qquad \text{with} \quad \Theta_z(t) = \int_0^t \Delta\omega(\tau)d\tau$$

The initial state is

$$\left|\psi_0\right\rangle = c_0\left(t_0\right)\left|0\right\rangle + c_1\left(t_0\right)\left|1\right\rangle$$

Thus by exploiting the definition of the $\hat{\sigma}_z$ operator $[\hat{\sigma}_z\left|0\right\rangle = -\left|0\right\rangle; \hat{\sigma}_z\left|1\right\rangle = +\left|1\right\rangle]$

$$\left|\psi'(t)\right\rangle = c_0\left(0\right)\exp\left[i\frac{\Theta_z}{2}t\right]\left|0\right\rangle + c_1\left(0\right)\exp\left[-i\frac{\Theta_z}{2}t\right]\left|1\right\rangle$$

which can be also put in the form

$$\left|\psi'(t)\right\rangle = \exp\left[i\frac{\Theta_z(t)}{2}t\right]\left\{c_0\left(t_0\right)\left|0\right\rangle + c_1\left(t_0\right)\exp\left[-i\Theta_z t\right]\left|1\right\rangle\right\}$$

Z-control gate

$$|\psi'(t)\rangle = \underbrace{\exp\left[i\frac{\Theta_z(t)}{2}t\right]c_0\left(t_0\right)}_{c_0'(t)}|0\rangle + \underbrace{\exp\left[i\frac{\Theta_z(t)}{2}t\right]\exp\left[-i\Theta_z t\right]c_1\left(t_0\right)}_{c_1'(t)}|1\rangle$$

$$\begin{cases} c_0'\left(t\right) = \exp\left[i\frac{\Theta_z(t)}{2}t\right]c_0\left(t_0\right) \\ c_1'\left(t\right) = \exp\left[i\frac{\Theta_z(t)}{2}\right]\exp\left[-i\Theta_z(t)\right]c_1\left(t_0\right) \end{cases}$$

which can be put in a matrix form

$$\mathbf{c}'(t) = -\exp\left[i\frac{\Theta_\mathbf{z}}{2}t\right]\mathbf{Z}_{\Theta_\mathbf{z}}\left(t\right)\mathbf{c}'\left(0\right)$$

where $\mathbf{Z}_{\Theta_z(t)} = \begin{vmatrix} -1 & 0 \\ 0 & -e^{-i\Theta_z(t)} \end{vmatrix}$ with $\Theta_z(t) = \int_0^t \Delta\omega(\tau)d\tau$

Z-control gate

$$\mathbf{c}'(t) = -\exp\left[i\frac{\Theta_\mathbf{z}}{2}t\right]\mathbf{Z}_{\Theta_\mathbf{z}}(t)\,\mathbf{c}'(0)$$

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- the term $\frac{\Delta\omega_d}{2}\hat{\sigma}_z$ in the Hamiltonian enables a rotation around the $\hat{z}$-axis of the Bloch sphere through the control of the qubit transition frequency $\omega_q(t)$.
- To implement a π-pulse on the z axis, one has to choose the parameters such that $\Theta_z(t) = \pi$:

$$\mathbf{Z}_\pi = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

X-control gate

Let us assume that we now apply a pulse at the qubit frequency $\omega_d = \omega_t$, so that $\Delta\omega = 0$, we have:

$$\begin{cases} i\dfrac{d\left|\psi'\right\rangle}{dt} = -s(t)\dfrac{\Omega}{2}\left(Q\hat{\sigma}_x + I\hat{\sigma}_y\right)\left|\psi'\right\rangle \\ \left|\psi'\left(t_0\right)\right\rangle = \left|\psi_0\right\rangle \end{cases}$$

- an *in-phase* pulse ($\gamma = 0$, I-component) corresponds to rotations around the x-axis
- an *out-of-phase* pulse ($\gamma = \pi/2$, Q-component) corresponds to rotations about the y-axis

$$\begin{cases} Q = 1 \\ I = 0 \end{cases}$$

$$j(t) = J_m s(t)\cos\left(\omega_d t\right)$$

X-control

$$i\frac{d\left|\psi'\right\rangle}{dt} = -s(t)\frac{\Omega}{2}\hat{\sigma}_x\left|\psi'\right\rangle$$

$$\left|\psi'\left(t_0\right)\right\rangle = \left|\psi_0\right\rangle = c_0(0)|0\rangle + c_1(0)|1\rangle$$

X-control gate

X-control

$$i\frac{d\left|\psi'\right\rangle}{dt} = -s(t)\frac{\Omega}{2}\hat{\sigma}_x\left|\psi'\right\rangle$$

$$\left|\psi'\left(t_0\right)\right\rangle = \left|\psi_0\right\rangle = c_0(0)|0\rangle + c_1(0)|1\rangle$$

$$\left|\psi'(t)\right\rangle = \exp\left[i\frac{\Theta_x}{2}\hat{\sigma}_x\right]\left|\psi_0\right\rangle$$

$$\Theta_x(t) = \Omega\int_{t_0}^{t} s(\tau)d\tau$$

- $\Theta_x(t)$ is the angle by which a state is rotated given the coupling frequency Ω, and the waveform envelope, $s(t)$.
- To implement a π-pulse on the x axis, one has to choose the parameters such that $\Theta_x(t) = \pi$ with a driving signal in quadrature with the qubit drive.

X-control gate

$$|\psi'(t)\rangle = \exp\left[i\frac{\Theta_x}{2}\hat{\sigma}_x\right]|\psi_0\rangle \qquad \Theta_x(t) = Q\Omega\int_{t_0}^{t} s(\tau)d\tau$$

The initial state: $|\psi_0\rangle = c_0(t_0)|0\rangle + c_1(t_0)|1\rangle$

$$|\psi'(t)\rangle = \left[\cos\left(\frac{\Theta_x}{2}\right)\hat{I} + i\sin\left(\frac{\Theta_x}{2}\right)\hat{\sigma}_x\right]\left[c_0(0)|0\rangle + c_1(0)|1\rangle\right]$$

X-control gate

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Thus by exploiting the definition of the $\hat{\sigma}_z$ operator $[\hat{\sigma}_x|0\rangle = |1\rangle\,;\hat{\sigma}_x|1\rangle = +|0\rangle]$

$$|\psi'(t)\rangle = \cos\left(\frac{\Theta_x}{2}\right)[c_0(0)|0\rangle + c_1(0)|1\rangle] + i\sin\left(\frac{\Theta_x}{2}\right)[c_0(0)|1\rangle + c_1(0)|0\rangle]$$

X-control gate

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Thus by exploiting the definition of the $\hat{\sigma}_z$ operator $[\hat{\sigma}_x|0\rangle = |1\rangle\,;\hat{\sigma}_x|1\rangle = +|0\rangle]$

$$|\psi'(t)\rangle = \cos\left(\frac{\Theta_x}{2}\right)\left[c_0(0)|0\rangle + c_1(0)|1\rangle\right] + i\sin\left(\frac{\Theta_x}{2}\right)\left[c_0(0)|1\rangle + c_1(0)|0\rangle\right]$$

$$|\psi'(t)\rangle = \underbrace{\left[\cos\left(\frac{\Theta_x}{2}\right)c_0(0) + i\sin\left(\frac{\Theta_x}{2}\right)c_1(0)\right]}_{c_0(t)}|0\rangle + \underbrace{\left[i\sin\left(\frac{\Theta_x}{2}\right)c_0(0) + \cos\left(\frac{\Theta_x}{2}\right)c_1(0)\right]}_{c_1}|0\rangle$$

X-control gate

$$|\psi'(t)\rangle = \underbrace{\left[\cos\left(\frac{\Theta_x}{2}\right)c_0(0) + i\sin\left(\frac{\Theta_x}{2}\right)c_1(0)\right]}_{c_0(t)}|0\rangle + \underbrace{\left[i\sin\left(\frac{\Theta_x}{2}\right)c_0(0) + \cos\left(\frac{\Theta_x}{2}\right)c_1(0)\right]}_{c_1}|0\rangle$$

X-control gate

$$|\psi'(t)\rangle = \underbrace{\left[\cos\left(\frac{\Theta_x}{2}\right) c_0(0) + i\sin\left(\frac{\Theta_x}{2}\right) c_1(0)\right]}_{c_0(t)} |0\rangle + \underbrace{\left[i\sin\left(\frac{\Theta_x}{2}\right) c_0(0) + \cos\left(\frac{\Theta_x}{2}\right) c_1(0)\right]}_{c_1} |0\rangle$$

$$\begin{cases} c_0'(t) = \cos\left(\frac{\Theta_x}{2}\right) c_0(0) + i\sin\left(\frac{\Theta_x}{2}\right) c_1(0) \\ c_1'(t) = i\sin\left(\frac{\Theta_x}{2}\right) c_0(0) + \cos\left(\frac{\Theta_x}{2}\right) c_1(0) \end{cases}$$

X-control gate

$$|\psi'(t)\rangle = \underbrace{\left[\cos\left(\frac{\Theta_x}{2}\right) c_0(0) + i\sin\left(\frac{\Theta_x}{2}\right) c_1(0)\right]}_{c_0(t)} |0\rangle + \underbrace{\left[i\sin\left(\frac{\Theta_x}{2}\right) c_0(0) + \cos\left(\frac{\Theta_x}{2}\right) c_1(0)\right]}_{c_1} |0\rangle$$

$$\begin{cases} c_0'(t) = \cos\left(\frac{\Theta_x}{2}\right) c_0(0) + i\sin\left(\frac{\Theta_x}{2}\right) c_1(0) \\ c_1'(t) = i\sin\left(\frac{\Theta_x}{2}\right) c_0(0) + \cos\left(\frac{\Theta_x}{2}\right) c_1(0) \end{cases}$$

$$\underbrace{\begin{pmatrix} c_0'(t) \\ c_1'(t) \end{pmatrix}}_{\mathbf{c}'(t)} = i \underbrace{\begin{pmatrix} -i\cos\left(\frac{\Theta_x}{2}\right) & \sin\left(\frac{\Theta_x}{2}\right) \\ \sin\left(\frac{\Theta_x}{2}\right) & -i\cos\left(\frac{\Theta_x}{2}\right) \end{pmatrix}}_{\mathbf{X}_{\Theta_x}} \underbrace{\begin{pmatrix} c_0(0) \\ c_1(0) \end{pmatrix}}_{\mathbf{c}_0(t)}$$

X-control gate

$$\mathbf{c}'(t) = i\mathbf{X}_{\Theta_\mathbf{z}}(t)\,\mathbf{c}'(0)$$

where $$\mathbf{X}_{\Theta_z(t)} = \begin{pmatrix} -i\cos\left(\frac{\Theta_x}{2}\right) & \sin\left(\frac{\Theta_x}{2}\right) \\ \sin\left(\frac{\Theta_x}{2}\right) & -i\cos\left(\frac{\Theta_x}{2}\right) \end{pmatrix} \qquad \Theta_x(t) = \Omega\int_{t_0}^{t} s(\tau)d\tau$$

- the term $-s(t)\frac{\Omega}{2}\frac{\Delta\omega_d}{2}\hat{\sigma}_x$ in the Hamiltonian enables a rotation around the $\hat{x}$-axis of the Bloch sphere through the control of the envelope signal $s(t)$.

X-control gate

$$\mathbf{c}'(t) = i\mathbf{X}_{\Theta_\mathbf{z}}(t)\,\mathbf{c}'(0)$$

where $\quad \mathbf{X}_{\Theta_z(t)} = \begin{pmatrix} -i\cos\left(\frac{\Theta_x}{2}\right) & \sin\left(\frac{\Theta_x}{2}\right) \\ \sin\left(\frac{\Theta_x}{2}\right) & -i\cos\left(\frac{\Theta_x}{2}\right) \end{pmatrix} \qquad \Theta_x(t) = \Omega\int_{t_0}^{t} s(\tau)d\tau$

- the term $-s(t)\frac{\Omega}{2}\frac{\Delta\omega_d}{2}\hat{\sigma}_x$ in the Hamiltonian enables a rotation around the $\hat{x}$-axis of the Bloch sphere through the control of the envelope signal $s(t)$.
- To implement a π-pulse on the x axis, one has to choose the parameters such that $\Theta_x(t) = \pi$ with a driving signal in quadrature with the qubit drive.

$$\mathbf{X}_\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Y-control gate

Let us assume that we now apply a pulse at the qubit frequency $\omega_d = \omega_t$, so that $\Delta\omega = 0$, we have:

$$\begin{cases} i\dfrac{d\left|\psi'\right\rangle}{dt} = -s(t)\dfrac{\Omega}{2}\left(Q\hat{\sigma}_x + I\hat{\sigma}_y\right)\left|\psi'\right\rangle \\ \left|\psi'\left(t_0\right)\right\rangle = \left|\psi_0\right\rangle \end{cases}$$

- an *in-phase* pulse ($\gamma = 0$, I-component) corresponds to rotations around the x-axis
- an *out-of-phase* pulse ($\gamma = \pi/2$, Q-component) corresponds to rotations about the y-axis

$$\begin{cases} Q = 0 \\ I = 1 \end{cases}$$

$$j(t) = J_m s(t) \sin\left(\omega_d t\right)$$

Y-control

$$i\frac{d\left|\psi'\right\rangle}{dt} = -s(t)\frac{\Omega}{2}\hat{\sigma}_y\left|\psi'\right\rangle$$

$$\left|\psi'\left(t_0\right)\right\rangle = \left|\psi_0\right\rangle = c_0(0)|0\rangle + c_1(0)|1\rangle$$

Y-control gate

$$\mathbf{c}'(t) = i\mathbf{Y}_{\Theta_y}(t)\,\mathbf{c}'(0)$$

where $$\mathbf{Y}_{\Theta_y(t)} = \begin{pmatrix} -i\cos\left(\frac{\Theta_y}{2}\right) & -i\sin\left(\frac{\Theta_y}{2}\right) \\ \sin\left(\frac{\Theta_y}{2}\right) & -i\cos\left(\frac{\Theta_y}{2}\right) \end{pmatrix} \qquad \Theta_y(t) = \Omega\int_{t_0}^{t} s(\tau)d\tau$$

- the term $-s(t)\frac{\Omega}{2}\frac{\Delta\omega_d}{2}\hat{\sigma}_y$ in the Hamiltonian enables a rotation around the y-axis of the Bloch sphere through the control of the envelope signal $s(t)$.

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- the term $-s(t)\frac{\Omega}{2}\frac{\Delta\omega_d}{2}\hat{\sigma}_y$ in the Hamiltonian enables a rotation around the y-axis of the Bloch sphere through the control of the envelope signal $s(t)$.
- To implement a π-pulse on the y axis, one has to choose the parameters such that $\Theta_x(t) = \pi$ with a driving signal in quadrature with the qubit drive.

$$\mathbf{Y}_{\pi} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Swap Operation

SWAP

$$|00\rangle \rightarrow |00\rangle$$
$$|10\rangle \rightarrow |01\rangle$$
$$|01\rangle \rightarrow |10\rangle$$
$$|11\rangle \rightarrow |11\rangle$$

$$\mathtt{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

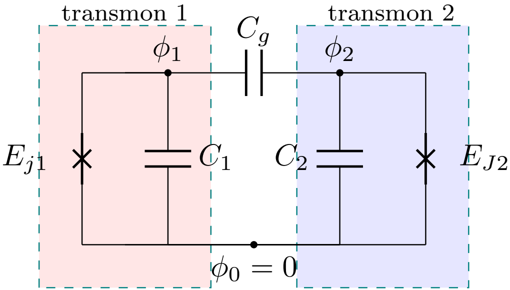
iSWAP

$$|00\rangle \rightarrow |00\rangle$$
$$|10\rangle \rightarrow i\,|01\rangle$$
$$|01\rangle \rightarrow i\,|10\rangle$$
$$|11\rangle \rightarrow |11\rangle$$

$$\mathtt{iSWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Two coupled transmons

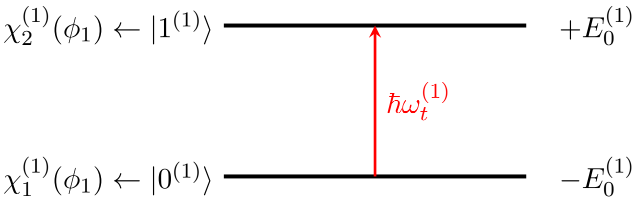
Hamiltonian in the weak-coupling regime



- Two degrees of freedom: fluxes ϕ_1 and ϕ_2.
- $\{\phi_1, Q_1\}$; $\{\phi_2, Q_2\}$ are conjugate variables.
- q_1, q_2 are the charges associated with the capacitors C_1, C_2. Note $q_1 \neq Q_1$, $q_2 \neq Q_2$.
- Weak coupling regime: $C_g \ll C_1, C_2$
- In this limit, we have $q_1 \approx Q_1$ and $q_2 \approx Q_2$

$$\hat{H} \cong \overbrace{\frac{\hat{q}_1^2}{2C_1} + E_{J1}\left[1 - \cos\left(\frac{2\pi\hat{\phi}_2}{\Phi_0}\right)\right]}^{\hat{H}_a^{(1)}} + \overbrace{\frac{\hat{q}_2^2}{2C_2} + E_{J2}\left[1 - \cos\left(\frac{2\pi\hat{\phi}_2}{\Phi_0}\right)\right]}^{\hat{H}_a^{(2)}} + \overbrace{\frac{C_g}{C_1C_2}\hat{q}_1\hat{q}_2}^{\hat{H}_{\text{int}}}$$

Two-level approximated model: Transmon 1

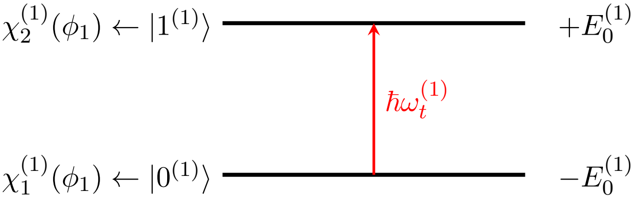


$$\omega_t^{(1)} = \sqrt{8E_{C1}|E_{J1}|} - \frac{E_{C1}}{\hbar}$$

$$\widehat{H}_a^{(1)} \cong \frac{\hbar\omega_t^{(1)}}{2}\hat{\sigma}_z^{(1)}$$

$\hat{\sigma}_z^{(1)}$ definition

$$\begin{cases} \hat{\sigma}_z^{(1)}|0^{(1)}\rangle = -|0^{(1)}\rangle \\ \hat{\sigma}_z^{(1)}|1^{(1)}\rangle = +|1^{(1)}\rangle \end{cases}$$



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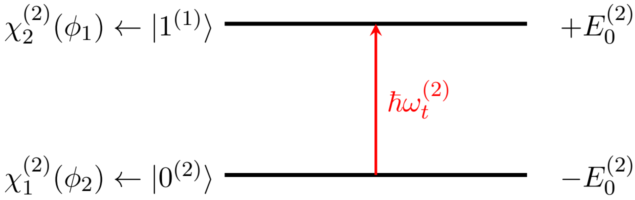
$$\begin{cases} \sigma_-^{(1)}|0^{(1)}\rangle = 0 \\ \sigma_-^{(1)}|1^{(1)}\rangle = |0^{(1)}\rangle \end{cases}$$

$\sigma_+^{(1)}$ definition

$$\begin{cases} \sigma_+^{(1)}|0^{(1)}\rangle = |1^{(1)}\rangle \\ \sigma_+^{(1)}|1^{(1)}\rangle = 0 \end{cases}$$

Two-level approximated model: Transmon 2



$$\omega_t^{(2)} = \sqrt{8E_{C2}|E_{J2}|} - \frac{E_2}{\hbar}$$

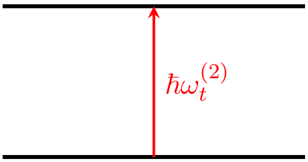
$$\widehat{H}_a^{(2)} \cong \frac{\hbar\omega_t^{(2)}}{2}\hat{\sigma}_z^{(2)}$$

$\hat{\sigma}_z^{(2)}$ definition

$$\begin{cases} \hat{\sigma}_z^{(2)}|0^{(2)}\rangle = -|0^{(2)}\rangle \\ \hat{\sigma}_z^{(2)}|1^{(2)}\rangle = +|1^{(2)}\rangle \end{cases}$$

Two-level approximated model: Transmon 2

$\chi_2^{(2)}(\phi_1) \leftarrow |1^{(1)}\rangle$ $+E_0^{(2)}$



$\chi_1^{(2)}(\phi_2) \leftarrow |0^{(2)}\rangle$ $-E_0^{(2)}$

$$\omega_t^{(2)} = \sqrt{8E_{C2}|E_{J2}|} - \frac{E_2}{\hbar}$$

$$\widehat{H}_a^{(2)} \cong \frac{\hbar\omega_t^{(2)}}{2}\hat{\sigma}_z^{(2)}$$

$\hat{\sigma}_z^{(2)}$ definition

$$\begin{cases} \hat{\sigma}_z^{(2)}|0^{(2)}\rangle = -|0^{(2)}\rangle \\ \hat{\sigma}_z^{(2)}|1^{(2)}\rangle = +|1^{(2)}\rangle \end{cases}$$

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$$\begin{cases} \sigma_-^{(2)}|0^{(2)}\rangle = 0 \\ \sigma_-^{(2)}|1^{(2)}\rangle = |0^{(2)}\rangle \end{cases}$$

$\sigma_+^{(1)}$ definition

$$\begin{cases} \sigma_+^{(2)}|0^{(2)}\rangle = |1^{(2)}\rangle \\ \sigma_+^{(2)}|1^{(2)}\rangle = 0 \end{cases}$$

Two-level approximated model: Interaction Terms

$$\widehat{H}_{\text{int}} = \frac{C_g}{C_1 C_2}\hat{q}_1\hat{q}_2$$

$$\widehat{H}_{\text{int}} = \hbar\Omega_{\text{int}}\,\hat{\sigma}_x^{(1)}\hat{\sigma}_x^{(2)}$$

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$$\begin{cases}\hat{\sigma}_x^{(1)}|0^{(1)}\rangle = |1^{(1)}\rangle \\ \hat{\sigma}_x^{(1)}|1^{(1)}\rangle = |0^{(1)}\rangle\end{cases}$$

$\hat{\sigma}_x^{(2)}$ definition

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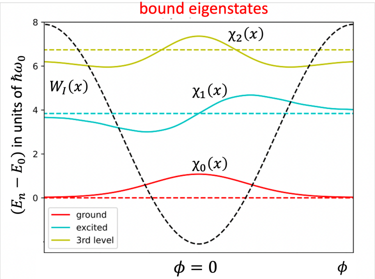
$$\begin{cases} \hat{\sigma}_x^{(2)}|0^{(2)}\rangle = |1^{(2)}\rangle \\ \hat{\sigma}_x^{(2)}|1^{(2)}\rangle = |0^{(2)}\rangle \end{cases}$$

$$\Omega_{int} = \frac{1}{\hbar}\frac{C_g}{C_1 C_2} q_{\mathsf{T}}^{(1)} q_{\mathsf{T}}^{(2)}$$

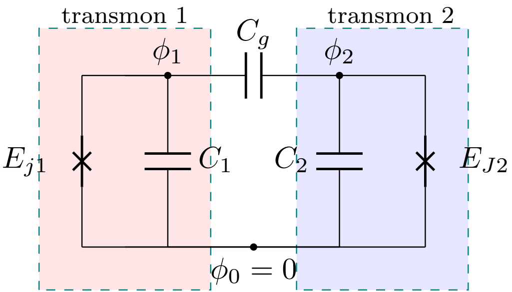
$$q_{\mathsf{T}}^{(1)} = \int_{-\Phi_0/2}^{+\Phi_0/2} \chi_1^{(1)} \frac{\hbar}{i}\frac{d\chi_0^{(1)}}{d\phi} d\phi$$

$$q_{\mathsf{T}}^{(2)} = \int_{-\Phi_0/2}^{+\Phi_0/2} \chi_1^{(2)} \frac{\hbar}{i}\frac{d\chi_0^{(2)}}{d\phi} d\phi$$



Circuit Hamiltonian in a two-level approximation

$$\hat{H} \cong \overbrace{\frac{\hat{q}_1^2}{2C_1} + E_{J1}\left[1 - \cos\left(\frac{2\pi\hat{\phi}_2}{\Phi_0}\right)\right]}^{\hat{H}_a^{(1)}} + \overbrace{\frac{\hat{q}_2^2}{2C_2} + E_{J2}\left[1 - \cos\left(\frac{2\pi\hat{\phi}_2}{\Phi_0}\right)\right]}^{\hat{H}_a^{(2)}} + \overbrace{\frac{C_g}{C_1C_2}\hat{q}_1\hat{q}_2}^{\hat{H}_{\text{int}}}$$



$$\hat{H} \cong +\frac{\hbar\omega_t^{(1)}}{2}\hat{\sigma}_z^{(1)} + \frac{\hbar\omega_t^{(2)}}{2}\hat{\sigma}_z^{(2)} + \hbar\Omega_{int}\hat{\sigma}_x^{(1)}\hat{\sigma}_x^{(2)}$$

$$\begin{cases} i\hbar\dfrac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle \\ |\psi(0)\rangle = |\psi_0\rangle \end{cases}$$

Tensor product between vector spaces

- Let V and U be two linear vector spaces, of dimension N_V and N_U respectively
- Let $\{|v_1\rangle, |v_2\rangle, \ldots\}$ be the basis of V
- Let $\{|u_1\rangle, |u_2\rangle, \ldots\}$ be the basis of V and U, respectively.
- Vectors and operators of these spaces are denoted with an index, (v) or (u), depending on whether they belong to V or U.

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- Vectors and operators of these spaces are denoted with an index, (v) or $(\ u\ )$, depending on whether they belong to V or U.

By definition, the vector space W is called the **tensor product** of V and U

$$W = V \otimes U$$

Tensor product between vector spaces

- the set of vectors $\{|v_m\rangle \otimes |u_n\rangle\,, m = 1, 2, \ldots; n = 1, 2, \ldots\}$ constitutes a basis for W;
- the tensor product is **distributive**

$$[\lambda|\varphi(v)\rangle] \otimes [|\chi(u)\rangle] = \lambda[|\varphi(v)\rangle \otimes |\chi(u)\rangle]$$
$$[|\varphi(v)\rangle] \otimes [\mu|\chi(u)\rangle] = \mu[|\varphi(v)\rangle \otimes |\chi(u)\rangle]$$

- the tensor product is **linear**

$$|\varphi(v)\rangle \otimes [\lambda_1 |\chi_1(u)\rangle + \lambda_2 |\chi_2(u)\rangle] = \lambda_1|\varphi(v)\rangle \otimes |\chi_1(u)\rangle + \lambda_2|\varphi(v)\rangle \otimes |\chi_2(u)\rangle$$
$$[\lambda_1 |\varphi_1(v)\rangle + \lambda_2 |\varphi_2(v)\rangle] \otimes [|\chi(u)\rangle] = \lambda_1 |\varphi_1(v)\rangle \otimes |\chi(u)\rangle + \lambda_2 |\varphi_2(v)\rangle \otimes |\chi(u)\rangle$$

- The scalar products in V and U permits us to define a scalar product in W as well. Let be

$$|\varphi(v), \chi(u)\rangle \equiv |\varphi(v)\rangle \otimes |\chi(u)\rangle \quad \text{and} \quad |\varphi'(v), \chi'(u)\rangle \equiv |\varphi'(v)\rangle \otimes |\chi'(u)\rangle\,.$$

Then $\langle\varphi'(v), \chi'(u) \mid \varphi(v), \chi(u)\rangle = \langle\varphi'(v) \mid \varphi(v)\rangle \langle\chi'(u) \mid \chi(u)\rangle$

Tensor product of operators

- We consider a linear operator $\mathcal{A}(v)$ defined in V.
- We introduce $\tilde{\mathcal{A}}(v)$ acting in W, which we call the extension of $\mathcal{A}(v)$ in W, defined in the following way:

$$\tilde{\mathcal{A}}(v)[|\varphi(v)\rangle \otimes |\chi(u)\rangle] = [\mathcal{A}(v)|\varphi(v)\rangle] \otimes |\chi(u)\rangle$$

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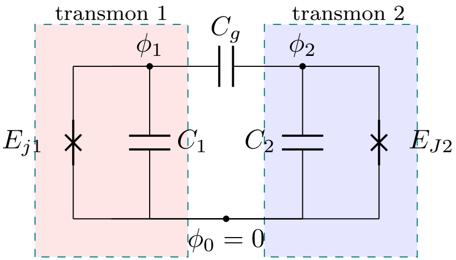
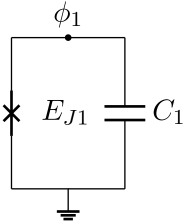
- Now let $\mathcal{A}(v)$ and $\mathcal{B}(u)$ be two linear operators acting respectively in V and U.
- Their tensor product $\mathcal{A}(v) \otimes \mathcal{B}(u)$ is the linear operator in W, such as

$$[\mathcal{A}(v) \otimes \mathcal{B}(u)][|\varphi(v)\rangle \otimes |\chi(u)\rangle] = [\mathcal{A}(v)|\varphi(v)\rangle] \otimes [\mathcal{B}(u)|\chi(u)\rangle]$$

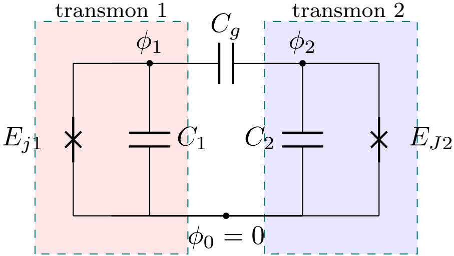
State space of two qubits



For $C_g \to 0$ the two qubits decouple



$$\left|\psi^{(1)}\right\rangle = c_0^{(1)}\left|0^{(1)}\right\rangle + c_1^{(1)}\left|1^{(1)}\right\rangle$$



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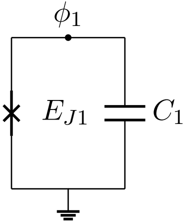

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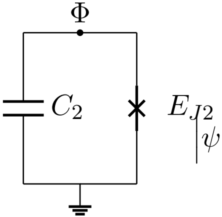
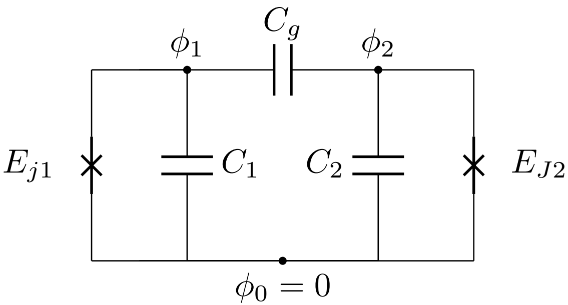


$$\left|\psi^{(1)}\right\rangle = c_0^{(2)}\left|0^{(2)}\right\rangle + c_1^{(2)}\left|1^{(2)}\right\rangle$$

State space of two qbits

A **basis** for the state space $S = S^{(1)} \otimes S^{(2)}$ is given by the set of orthonormal kets



- $|00\rangle = \left|0^{(1)}\right\rangle \otimes \left|0^{(2)}\right\rangle$;
- $|01\rangle = \left|0^{(1)}\right\rangle \otimes \left|1^{(2)}\right\rangle$
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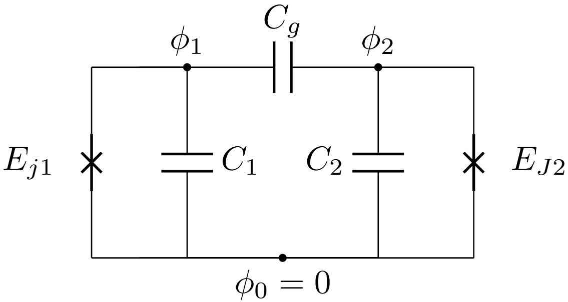

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$$|\psi(t)\rangle = c_{00}(t)|00\rangle + c_{01}(t)|01\rangle + c_{10}(t)|10\rangle + c_{11}(t)|11\rangle$$

$$1 = |c_{00}|^2 + |c_{01}|^2 + |c_{10}|^2 + |c_{11}|^2$$

Schrödinger equation in the rotating frame

At $t = 0$ the state of the quantum circuit is $\psi_0 = |\psi(0)\rangle$

$$|\psi(0)\rangle = c_{00}(t)|00\rangle + c_{01}(0)|01\rangle + c_{10}(t)|10\rangle + c_{11}(0)|11\rangle$$

To find the state of the circuit $|\psi(t)\rangle$ at time t, we have to solve the Schrödinger equation

$$\begin{cases} i\hbar\dfrac{d}{dt}|\psi\rangle = \widehat{H}|\psi\rangle \\ |\psi\,(0)\rangle = |\psi_0\rangle \end{cases}$$

$$\widehat{H} = +\frac{\hbar\omega_t}{2}\hat{\sigma}_z^{(1)} \otimes \hat{I}^{(2)} + \hat{I}^{(1)} \otimes \frac{\hbar\omega_t}{2}\hat{\sigma}_z^{(2)} + \hbar\Omega_{int}\hat{\sigma}_x^{(1)} \otimes \hat{\sigma}_x^{(2)}$$

Variable transformation: rotating frame

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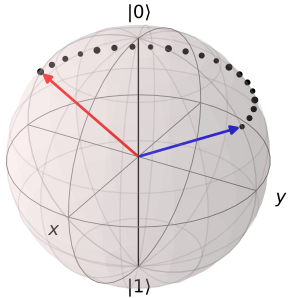
- At time t, the Bloch sphere representation of each qubit is rotated clockwise about the z-axis by an angle $\alpha(t) = +\omega_t t$ with respect to the Bloch sphere representation of $|\psi\rangle$



$$\widehat{R}_z^{(h)}(\alpha(t)) = -e^{i\alpha/2}\mathbf{Z}_\alpha^{(h)} = \exp\left(-i\alpha(t)\frac{\hat{\sigma}_z^{(h)}}{2}\right) \qquad \alpha(t) = -\omega_t t \qquad h \in \{1,2\}$$

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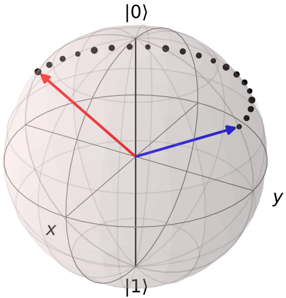

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$$\hat{R}(t) = \widehat{R}_z^{(1)}(\alpha(t)) \otimes \widehat{R}_z^{(2)}(\alpha(t)) = \exp\left[i\omega_t\,(t-t_0)\,\hat{\sigma}_z^{(1)}/2\right] \otimes \exp\left[i\omega_t\,(t-t_0)\,\hat{\sigma}_z^{(2)}/2\right]$$

Coupled Qubits in the Rotating Wave Approximation

- To find the state of the circuit $|\psi(t)\rangle$ at time t, we have to solve the Schrödinger equation

$$\begin{cases} i\hbar \dfrac{d}{dt} |\psi'\rangle = \widehat{H}'_{int} |\psi\rangle \\ |\psi'(0)\rangle = |\psi_0\rangle \end{cases}$$

$$\hat{H}' = \hbar\Omega_{int} \left[\sigma_-^{(1)} \otimes \sigma_+^{(2)} + \sigma_+^{(1)} \otimes \sigma_-^{(2)} + \sigma_-^{(1)} \otimes \sigma_-^{(2)} e^{-i2\omega_t t} + \sigma_+^{(1)} \otimes \sigma_+^{(2)} e^{+i2\omega_t t} \right]$$

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Disregarding the oscillatin terms at frequency $2\omega_t$ we obtain (rotating wave approximation):

$$\hat{H}' = \hbar\Omega_{int}\left[\sigma_-^{(1)} \otimes \sigma_+^{(2)} + \sigma_+^{(1)} \otimes \sigma_-^{(2)} + \cancel{\sigma_-^{(1)} \otimes \sigma_-^{(2)} e^{-i2\omega_t t}} + \cancel{\sigma_+^{(1)} \otimes \sigma_+^{(2)} e^{+i2\omega_t t}}\right]$$

Rotating Wave Approximation

$$\begin{cases} i\hbar \dfrac{d}{dt}\left|\psi'\right\rangle \cong \hbar\Omega_{\text{int}}\left(\sigma_-^{(1)} \otimes \sigma_+^{(2)} + \sigma_+^{(1)} \otimes \sigma_-^{(2)}\right)\left|\psi'\right\rangle \\ \left|\psi'\left(t_0\right)\right\rangle = \left|\psi_0\right\rangle \end{cases}$$

- We **expand the unknown state** $|\psi'(t)\rangle$ in terms of the four basis states of $\mathbb{S}$:

$$|\psi'(t)\rangle = c_{00}(t)|00\rangle + c_{01}(t)|01\rangle + c_{10}(t)|10\rangle + c_{11}(t)|11\rangle$$

- Our goal is to determine the 4 coefficients, which are function of time and subjected to the normalization condition.

Rotating Wave Approximation

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- We **expand the unknown state** $|\psi'(t)\rangle$ in terms of the four basis states of $\mathbb{S}$:

$$|\psi'(t)\rangle = c_{00}(t)|00\rangle + c_{01}(t)|01\rangle + c_{10}(t)|10\rangle + c_{11}(t)|11\rangle$$

- Our goal is to determine the 4 coefficients, which are function of time and subjected to the normalization condition.
- We substitute this expansion in the Schrödinger equation

Rotating Wave Approximation

$$\begin{cases} i\hbar \dfrac{d}{dt} |\psi'\rangle \cong \hbar\Omega_{\text{int}} \left(\sigma_-^{(1)} \otimes \sigma_+^{(2)} + \sigma_+^{(1)} \otimes \sigma_-^{(2)}\right) |\psi'\rangle \\ |\psi'(t_0)\rangle = |\psi_0\rangle \end{cases}$$

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- Our goal is to determine the 4 coefficients, which are function of time and subjected to the normalization condition.
- We substitute this expansion in the Schrödinger equation
- The operators $\sigma_+^{(1)} \otimes \sigma_-^{(2)}$ and $\sigma_-^{(1)} \otimes \sigma_+^{(2)}$ will q-bitwise act on the four basis states accordingly to the definition of $\sigma_\pm^{(1)}$ and $\sigma_\pm^{(2)}$

Coordinate evolution

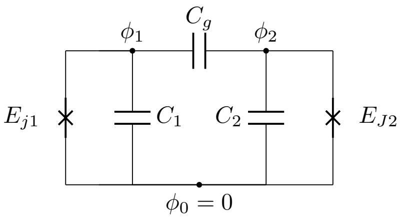
- We project the resulting equation along each of the four basis states of $\mathbb{S}$, obtaining:

$$\begin{cases} \dot{c}_{00}(t) = 0 \\ \dot{c}_{01}(t) = -i\Omega_{int}\, c_{10}(t) \\ \dot{c}_{10}(t) = -i\Omega_{int}\, c_{01}(t) \\ \dot{c}_{11}(t) = 0 \end{cases} \quad \begin{cases} \dot{c}_{00}(t) = \text{const} \\ \ddot{c}_{01}(t) = -\Omega_{int}^2 c_{01}(t) \\ c_{10}(t) = i/\Omega_{int}\dot{c}_{01}(t) \\ \dot{c}_{11}(t) = \text{const} \end{cases} \quad \begin{cases} \dot{c}_{00}(t) = \text{const} \\ c_{01}(t) = +K_+ e^{+i\Omega_{int}t} + K_- e^{-i\Omega_{int}t} \\ c_{10}(t) = -K_+ e^{+i\Omega_{int}t} + K_- e^{-i\Omega_{int}t} \\ \dot{c}_{11}(t) = \text{const} \end{cases}$$

- We enforce initial conditions at initial time.

$$\begin{cases} c_{00}(t=0) = c_{00}(0), \\ c_{01}(t=0) = c_{01}(0), \\ c_{10}(t=0) = c_{10}(0), \\ c_{11}(t=0) = c_{11}(0), \end{cases} \quad \begin{cases} \dot{c}_{00}(t) = c_{00}(0) \\ c_{01}(0) = +K_+ + K_- \\ c_{10}(0) = -K_+ + K_- \\ \dot{c}_{11}(t) = c_{00}(0) \end{cases} \quad \begin{cases} 2K_- = c_{01}(0) + c_{10}(0) \\ 2K_+ = c_{01}(0) - c_{10}(0) \end{cases}$$

Coordinate evolution



The two coupled qubits, initially in the state:

$$|\psi'(0)\rangle = c_{00}(0)|00\rangle + c_{01}(0)|01\rangle + c_{10}(0)|10\rangle + c_{11}(0)|11\rangle$$

after a time t their state (in the rotating frame) is:

$$|\psi'(t)\rangle = c_{00}(t)|00\rangle + c_{01}(t)|01\rangle + c_{10}(t)|10\rangle + c_{11}(t)|11\rangle$$

where the coefficients are the following time-functions

$$\begin{cases} c_{00}(t) = c_{00}(0) \\ c_{01}(t) = c_{01}(0)\cos\left(\Omega_{int} t\right) - i c_{10}(0)\sin\left(\Omega_{\text{int}}\, t\right) \\ c_{10}(t) = -i c_{01}(0)\sin\left(\Omega_{int} t\right) + c_{10}(0)\cos\left(\Omega_{\text{int}}\, t\right) \\ c_{11}(t) = c_{11}(0) \end{cases}$$

Matrix representation of the evolution operator

The coefficient can be also represented in a matrix form as:

$$\mathbf{c}(t) = \begin{vmatrix} c_{00}(t) \\ c_{01}(t) \\ c_{10}(t) \\ c_{11}(t) \end{vmatrix} \qquad \mathbf{c}(t) = \mathbf{U}(t)\mathbf{c}(0) \qquad \mathbf{U}(t) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\left(\Omega_{int} t\right) & -i\sin\left(\Omega_{int} t\right) & 0 \\ 0 & -i\sin\left(\Omega_{int} t\right) & \cos\left(\Omega_{\text{int}}\, t\right) & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

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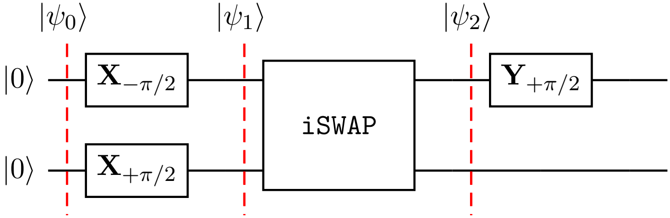
The two qubits are tuned in resonance, $\omega_t^{(1)} = \omega_t^{(2)}$ for a time interval Δt such that $\Omega_{int}\Delta t = \frac{\pi}{2}$,

$$\text{iSWAP} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Bell State Generation



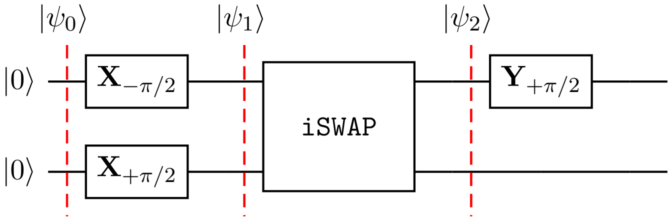
$$\mathbf{X}_{\Theta} = \begin{vmatrix} -i\cos\left(\frac{\Theta_x}{2}\right) & \sin\left(\frac{\Theta_x}{2}\right) \\ \sin\left(\frac{\Theta_x}{2}\right) & -i\cos\left(\frac{\Theta_x}{2}\right) \end{vmatrix};$$

$$\mathbf{X}_{\pm\pi/2} = \frac{1}{\sqrt{2}}\begin{vmatrix} -i & \pm 1 \\ \pm 1 & -i \end{vmatrix};$$

$$\mathbf{Y}_{\Theta} = -i\begin{vmatrix} \cos\left(\frac{\Theta_y}{2}\right) & \sin\left(\frac{\Theta_y}{2}\right) \\ -\sin\left(\frac{\Theta_y}{2}\right) & \cos\left(\frac{\Theta_y}{2}\right) \end{vmatrix}$$

$$\mathbf{Y}_{\pi/2} = -\frac{i}{\sqrt{2}}\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

Bell State Generation

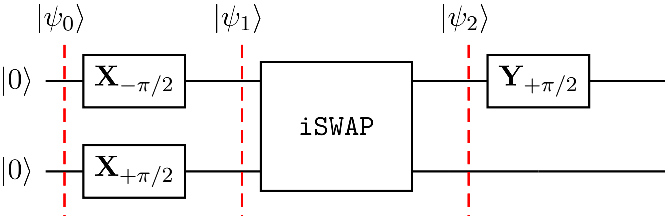


$$\mathbf{X}_{\pm\pi/2} = \frac{1}{\sqrt{2}} \begin{vmatrix} -i & \pm 1 \\ \pm 1 & -i \end{vmatrix}; \qquad \texttt{iSWAP} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \qquad \mathbf{Y}_{\Theta} = -\frac{i}{\sqrt{2}} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$\mathbf{X}_{-\pi/2} \otimes \mathbf{X}_{+\pi/2} = \frac{1}{2} \begin{vmatrix} -1 & -i & +i & -1 \\ -i & -1 & -1 & +i \\ +i & -1 & -1 & -i \\ -1 & +i & -i & -1 \end{vmatrix} \qquad \mathbf{Y}_{\pi/2} \otimes \mathbf{I} = -\frac{i}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

Bell State Generation

$$|\psi_0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

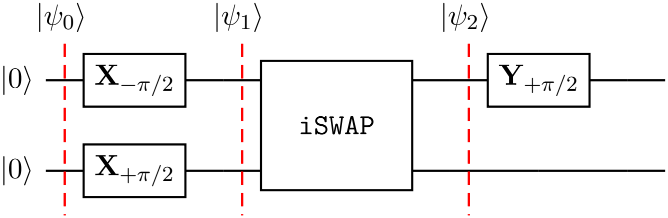


$$\mathbf{X}_{-\pi/2} \otimes \mathbf{X}_{+\pi/2} = \frac{1}{2} \begin{vmatrix} -1 & -i & +i & -1 \\ -i & -1 & -1 & +i \\ +i & -1 & -1 & -i \\ -1 & +i & -i & -1 \end{vmatrix}$$

$$|\psi_1\rangle = \mathbf{X}_{-\pi/2} \otimes \mathbf{X}_{+\pi/2} \, |\psi_0\rangle = \frac{1}{2} \begin{pmatrix} -1 \\ -i \\ +i \\ -1 \end{pmatrix}$$

Bell State Generation

$$|\psi_1\rangle = \frac{1}{2}\begin{pmatrix} -1 \\ i \\ -i \\ -1 \end{pmatrix}$$

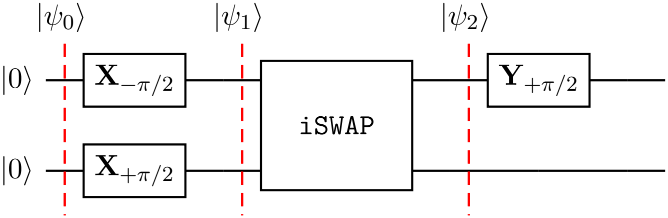


$$\texttt{iSWAP} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$|\psi_2\rangle = \texttt{iSWAP}\,|\psi_1\rangle = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

Bell State Generation

$$|\psi_2\rangle = -\frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$



$$\mathbf{Y}_{\pi/2} \otimes \mathbf{I} = -\frac{i}{\sqrt{2}}\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{vmatrix}$$

$$|\psi_3\rangle = \mathbf{Y}_{\pi/2} \otimes \mathbf{I}\,|\psi_2\rangle = \frac{i}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$